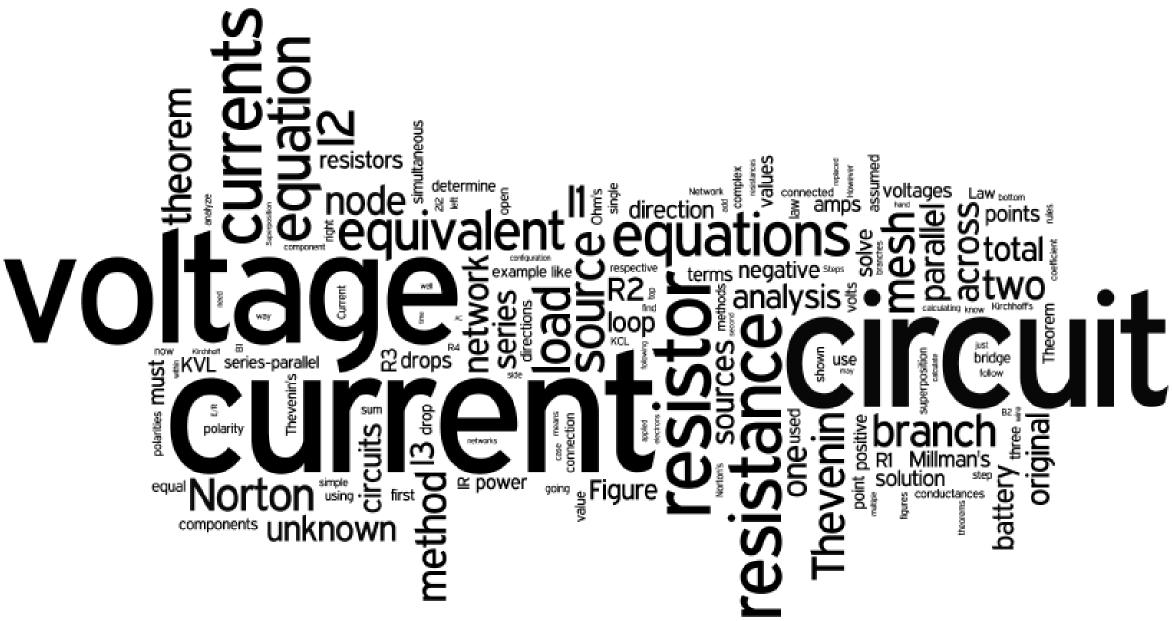
9 Network Analysis



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Network Analysis

Generally speaking, **network analysis** is any structured technique used to mathematically analyze a network of interconnected components, or circuit. Quite often the technician will encounter circuits containing multiple sources of power or component configurations which defy simplification by series- parallel analysis techniques. In those cases, he or she will be forced to use other means. This chapter presents a few techniques useful in analyzing such complex circuits.

To illustrate how even a simple circuit can defy analysis by breakdown into series and parallel portions, take start with the series-parallel circuit shown in Figure 1.

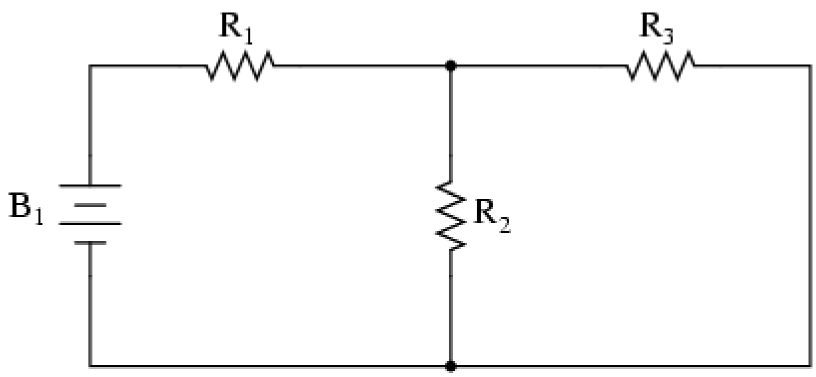

Figure 1: Series-Parallel Circuit

To analyze the above circuit, one would first find the equivalent of R_2 and R_3 in parallel, and then add R_1 in series to arrive at a total resistance. Then, taking the voltage of battery B_1 with that total circuit resistance, the total current could be calculated through the use of Ohm's Law (I=E/R), then that current figure used to calculate voltage drops in the circuit. However, as shown in Figure 2, the addition of just one more battery could change all of that.

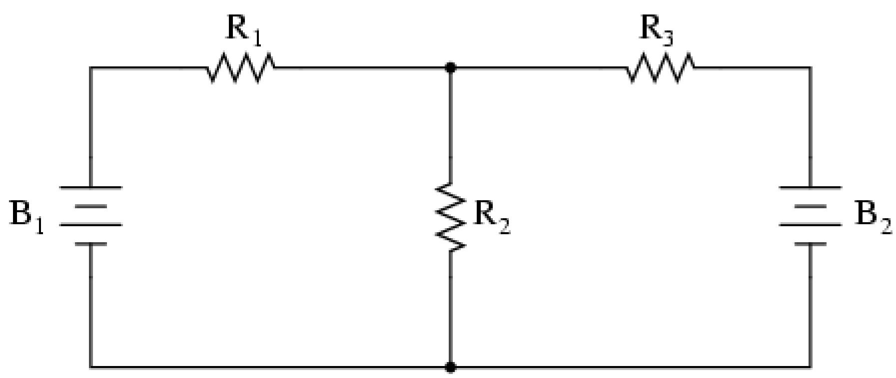

Figure 2: Series-Parallel Circuit with two voltage sources.

Resistors R_2 and R_3 are no longer in parallel with each other, because B_2 has been inserted into R_3's branch of the circuit. Upon closer inspection, it appears there are *no* two resistors in this circuit directly in series or parallel with each other. This is the crux of our problem: in series-parallel analysis, we started off by identifying sets of resistors that *were* directly in series or parallel with each other, reducing them to single equivalent resistances. If there are no resistors in a simple series or parallel configuration with each other, then what can we do?

It should be clear that this seemingly simple circuit, with only three resistors, is impossible to reduce as a combination of simple series and simple parallel sections: it is something different altogether. However, this is not the only type of circuit defying series-parallel analysis.

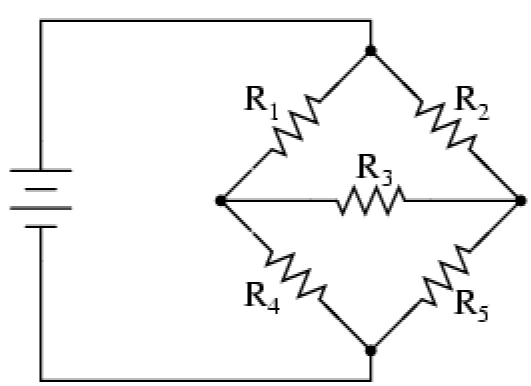

Figure 3: Unbalanced Bridge Circuit

Figure 3 shows an unbalanced bridge. If it were balanced, there would be zero current through R_3, and it could be approached as a series-parallel combination circuit (R_1R_4 // R_2R_5). However, any current through R_3 makes a series-parallel analysis impossible. R_1 is not in series with R_4 because there's another path for electrons to flow through R_3. Neither is R_2 in series with R_5 for the same reason. Likewise, R_1 is not in parallel with R_2 because R_3 is separating their bottom leads. Neither is R_4 in parallel with R_5.

Although it might not be apparent at this point, the heart of the problem is the existence of multiple unknown quantities. At least in a series-parallel combination circuit, there was a way to find total resistance and total voltage, leaving total current as a single unknown value to calculate (and then that current was used to satisfy previously unknown variables in the reduction process until the entire circuit could be analyzed). With these problems, more than one parameter (variable) is unknown at the most basic level of circuit simplification.

With the two-battery circuit, there is no way to arrive at a value for "total resistance," because there are *two* sources of power to provide voltage and current (we would need *two* "total" resistances in order to proceed with any Ohm's Law calculations). With the unbalanced bridge circuit, there is such a thing as total resistance across the one battery (paving the way for a calculation of total current), but that total current immediately splits up into unknown proportions at each end of the bridge, so no further Ohm's Law calculations for voltage (E=IR) can be carried out.

So what can we do when we're faced with multiple unknowns in a circuit? The answer is initially found in a mathematical process known as **simultaneous equations or systems of equations**, whereby multiple unknown variables are solved by relating them to each other in multiple equations. In a scenario with only one unknown, there only needs to be a single equation to solve for the single unknown, as shown in the list of Ohm's law derivations below.

$$\mathbf{E} = I\,R \quad (\mathbf{E} \text{ is unknown; } I \text{ and } R \text{ are known})$$

. . . or . . .

$$\mathbf{I} = \frac{E}{R} \quad (\mathbf{I} \text{ is unknown; } E \text{ and } R \text{ are known})$$

. . . or . . .

$$\mathbf{R} = \frac{E}{I} \quad (\mathbf{R} \text{ is unknown; } E \text{ and } I \text{ are known})$$

This module discusses various techniques of network analysis that can be used to solve complex circuits. First, there are several different methods using the Ohm's and Kirchhoff's laws, such as the branch and mesh current method, and the node voltage method. Finally, network theorems will be introduced. Network theorems, such as Millman's, Superposition, Thevenin's, and Norton's theorems provide the framework necessary for more specific problem solving techniques

Branch Current Method

The first and most straightforward network analysis technique is called the **branch current method**. In this method, we assume directions of currents in a network, and then write equations describing their relationships to each other through Kirchhoff's and Ohm's laws. Once we have one equation for every unknown current, we can solve the simultaneous equations and determine all currents, and therefore all voltage drops in the network.

The first step is to choose a node (junction of wires) in the circuit to use as a point of reference for our unknown currents. Figure 4 indicates the chosen node.

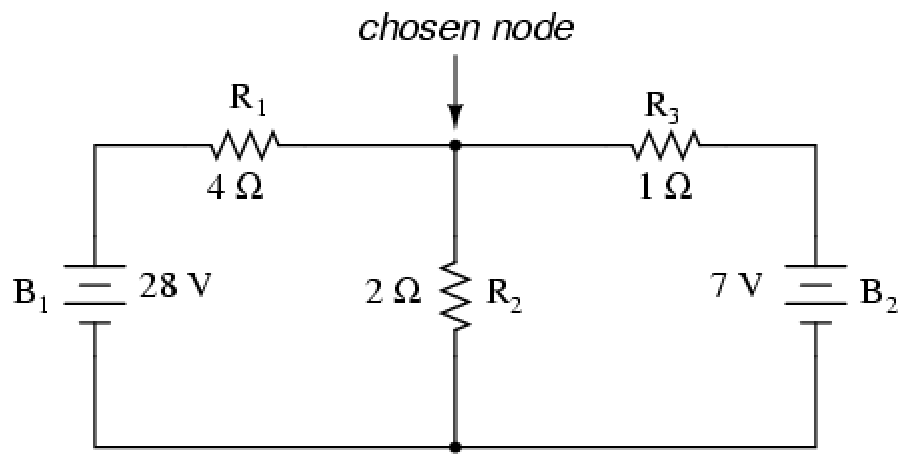

Figure 4: Choose a starting node for analysis.

At this node, assume the direction of the current flowing in to and out of the node and label the three currents as I_1, I_2, and I_3, as shown in Figure 5 . Be aware that the current are speculative at this point. Fortunately, if it turns out that any of our guesses were wrong, we will know when we mathematically solve for the currents (any “wrong” current directions will show up as negative numbers in our solution).

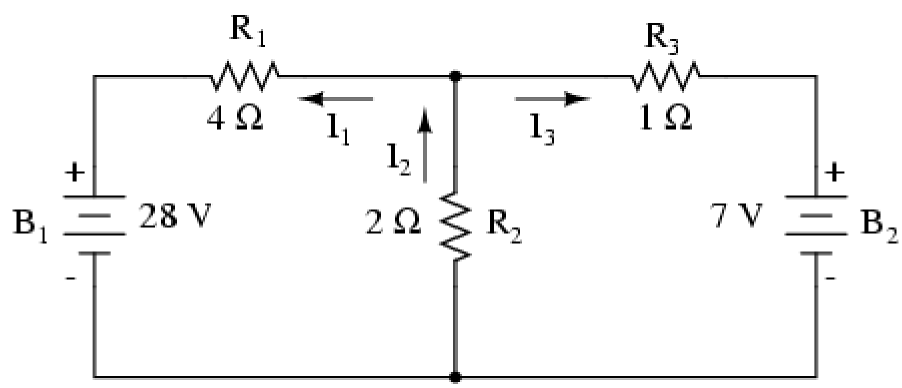

Figure 5: Node currents labeled.

Kirchhoff's current law (KCL) tells us that the algebraic sum of currents entering and exiting a node must equal zero, so we can relate these three currents (I_1, I_2, and I_3) to each other in a single equation. Currents *entering* the node will be labeled as positive, while currents *exiting* the node will be labeled as negative: $\mathbf{-I_1 + I_2 - I_3 = 0}$.

The next step is to label all voltage drop polarities across resistors according to the assumed directions of the currents (Figure 6). Remember that the "upstream" end of a resistor will always be negative, and the "downstream" end of a resistor positive with respect to each other, since electrons are negatively charged.

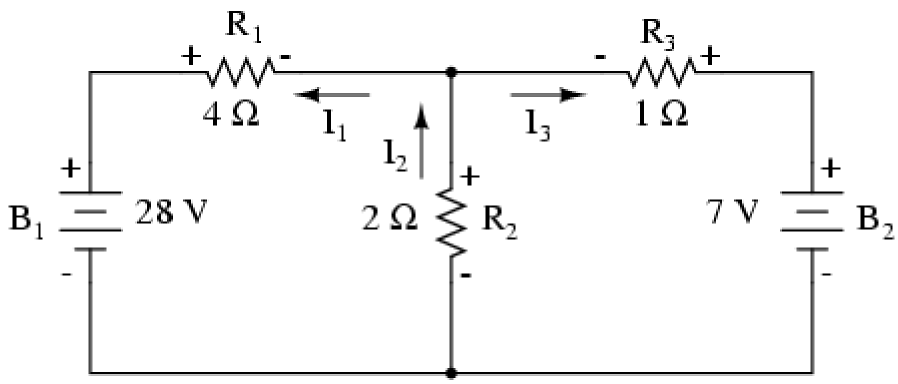

Figure 6: Label polarities of voltage drops

The battery polarities, of course, remain as they were according to the orientation at which they were drawn. Remember, the short line indicates the negative (or return) side of the battery and the long line indicates the positive end. It is acceptable if the polarity of a resistor's voltage drop doesn't match with the polarity of the nearest battery as long as the resistor voltage polarity is correctly based on the assumed direction of current through it. In some cases we may discover that current will be forced *backwards* through a battery, causing this very effect. The important thing to remember here is to base all your resistor polarities and subsequent calculations on the assumed directions of current(s). As stated earlier, if your assumption happens to be incorrect, it will be apparent once the equations have been solved (by means of a negative solution). The **magnitude** of the solution, however, will still be correct.

Kirchhoff's voltage law (KVL) states that the algebraic sum of all voltages in a loop must equal zero. Therefore, we can create more equations with current terms (I_1, I_2, and I_3) for our simultaneous equations. To obtain a KVL equation, record the voltage drops in each loop of the circuit. For the left loop, start at the upper-left corner and move counter-clockwise around the loop. Upon completion, your equation should look like this: $\mathbf{-28 + E_{R2} + E_{R1} = 0.}$

At this point the voltages across R_1 or R_2 are unknown, but according to KVL, we *do* know that all three voltages must algebraically add to zero, so the equation is true. We can go a step further and express the unknown voltages as the product of the corresponding unknown currents (I_1 and I_2) and their respective resistors, following Ohm's Law (E=IR):

Start with: $\mathbf{-28 + E_{R2} + E_{R1} = 0}$

Using Ohm's Law (E = IR), substitute IR for E in the KVL equation: $\mathbf{-28 + I_2R_2 + I_1R_1 = 0}$

Since we know what the values of all the resistors are in ohms, we can just substitute those figures into the equation to simplify things a bit: $\mathbf{-28 + 2I_2 + 4I_1 = 0}$

You might be wondering why we went through all the trouble of manipulating this equation from its initial form ($\mathbf{-28 + E_{R2} + E_{R1} = 0}$). After all, the last two terms are still unknown, so what advantage is there to expressing them in terms of unknown voltages or as unknown currents? The purpose in doing this is to get the KVL equation expressed using the *same unknown variables* as the KCL equation, as this is a necessary requirement for any simultaneous equation solution method. To solve for three unknown currents (I_1, I_2, and I_3), we must have three equations relating these three *currents* together.

Applying the same steps to the right loop of the circuit, starting at the chosen node and moving counter-clockwise, we get another KVL equation: $\mathbf{-E_{R2} + 7 - E_{R3} = 0.}$

Knowing now that the voltage across each resistor should be expressed as the product of the corresponding current and the resistance of each resistor, we can re-write the equation as such: $\mathbf{-2I_2 + 7 - I_3 = 0}$

Now we have a mathematical system of three equations (one KCL equation and two KVL equations) and three unknowns:

$-I_1 + I_2 - I_3 = 0$ **Kirchhoff's Current Law**

$-28 + 2I_2 + 4I_1 = 0$ **Kirchhoff's Voltage Law**

$-2I_2 + 7 - I_3 = 0$ **Kirchhoff's Voltage Law**

For some methods of solution, it is helpful to express each unknown term in each equation, with any constant value to the right of the equal sign, and with any "unity" terms expressed with an explicit coefficient of 1. Re-writing the equations again, we have:

$-1I_1 + 1I_2 - 1I_3 = 0$ **Kirchhoff's Current Law**

$4I_1 + 2I_2 + 0I_3 = 28$ **Kirchhoff's Voltage Law**

$0I_1 - 2I_2 - 1I_3 = -7$ **Kirchhoff's Voltage Law**

Using whatever solution techniques are available to us, we should arrive at a solution for the three unknown current values:

I_1 = 5A
I_2 = 4A
I_3 = -1A

Solving Simultaneous Equations

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Therefore, I_1 is 5 amps, I_2 is 4 amps, and I_3 is a negative 1 amp. But what does "negative" current mean? In this case, it means that our *assumed* direction for I_3 was opposite of its *real* direction. Going back to our original circuit, we can re-draw the circuit to be in line with what we have determined (Figure 7).

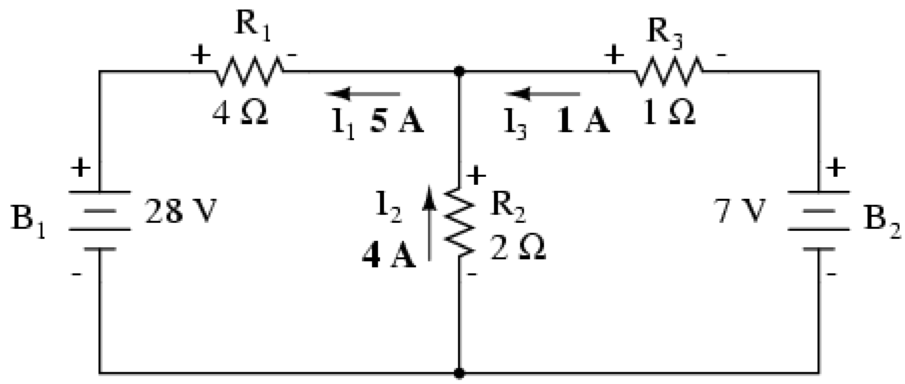


Figure 7: Redrawn with correct polarities.

Notice how current is being pushed backwards through battery B_2 (electrons flowing from positive side) due to the higher voltage of battery B_1, whose current is flowing from the negative side. Despite the fact that battery B_2's polarity is trying to push electrons down in that branch of the circuit, electrons are being forced backwards through it due to the superior voltage of battery B_1. Does this mean that the stronger battery will always overpower the weaker battery and force the current through it backwards? No, because it depends on both batteries' relative voltages ***and*** the resistor values in the circuit. The only way to determine what is going on is to take the time to mathematically analyze the network.

Now that we know the magnitude of all currents in this circuit, we can calculate voltage drops across all resistors with Ohm's Law (E=IR):

$$\mathbf{E_{R1} = I_1R_1 = (5\ A)\ (4\ \Omega) = 20\ V}$$

$$\mathbf{E_{R2} = I_2R_2 = (4\ A)\ (2\ \Omega) = 8\ V}$$

$$\mathbf{E_{R3} = I_3R_3 = (1\ A)\ (1\ \Omega) = 1\ V}$$

Steps to follow for the "Branch Current" method of analysis:

- Choose a node and assume directions of currents.
- Write a KCL equation relating currents at the node.

- Label resistor voltage drop polarities based on assumed currents.
- Write KVL equations for each loop of the circuit, substituting the product IR for E in each resistor term of the equations.
- Solve for unknown branch currents (simultaneous equations).
- If any solution is negative, then the assumed direction of current for that solution is wrong!
- Solve for voltage drops across all resistors (E=IR).

Mesh Current Method

The **mesh current method** also known as the **loop current method** is quite similar to the branch current method in that it uses simultaneous equations, Kirchhoff's voltage law, and Ohm's law to determine unknown currents in a network. It differs from the branch current method in that it does *not* use Kirchhoff's current law, and it is usually able to solve a circuit with less unknown variables and less simultaneous equations, which is especially nice if you are forced to solve without a calculator.

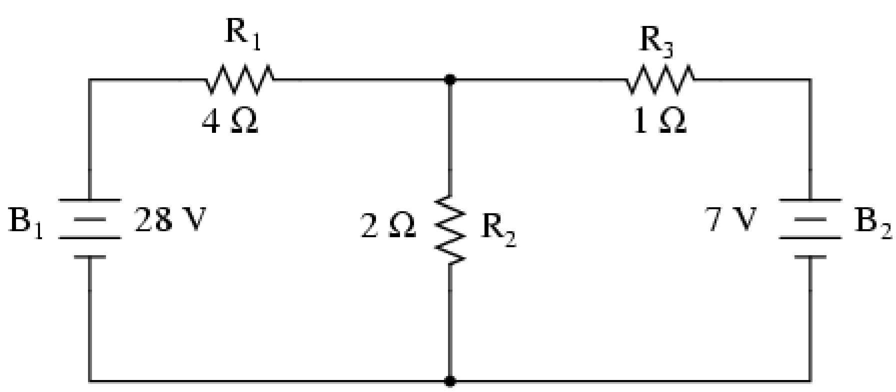


Figure 8: Series-Parallel Circuit to be analyzed.

The circuit (Figure 8) used for this example is the same as the one used for the branch current method. The first step in the mesh current method is to identify loops within the circuit encompassing all components. In our example circuit, the loop formed by B_1, R_1, and R_2 will be the first while the loop formed by B_2, R_2, and R_3 will be the second (Figure 9).

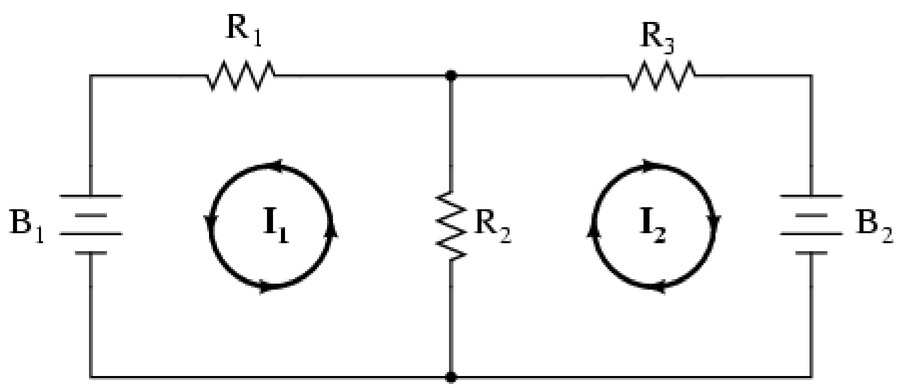


Figure 9: Identify loops in each part of the mesh.

The choice of each current's direction is entirely arbitrary, just as in the branch current method, but the resulting equations are easier to solve if the currents are going the same direction through intersecting components. In the figure above, note how currents I_1 and I_2 are both flowing up through resistor R_2, where they mesh, or intersect. As with the branch current method, if the assumed direction of a mesh current is wrong, the answer for that current will have a negative value.

The next step is to label all voltage drop polarities across resistors according to the assumed directions of the mesh currents as shown in Figure 10. Remember that the “upstream” end of a resistor will always be negative, and the “downstream” end of a resistor positive with respect to each other, since electrons are negatively charged. The battery polarities, of course, are dictated by their symbol orientations in the diagram, and may or may not “agree” with the resistor polarities (assumed current directions).

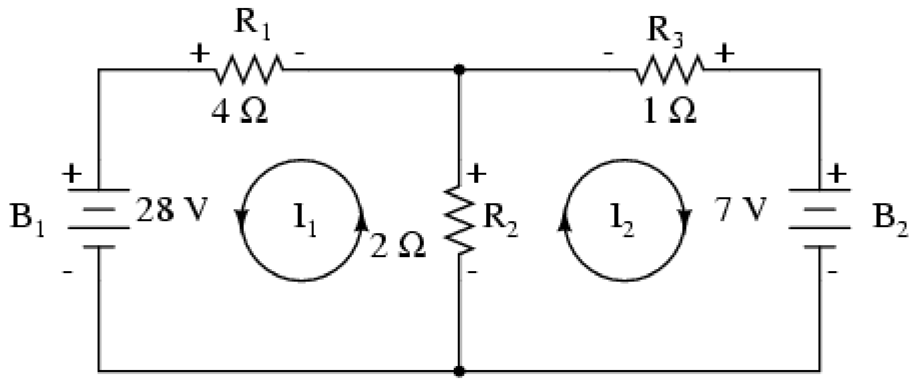


Figure 10: Label voltage drop polarities.

Using Kirchhoff's voltage law, we can now step around each of these loops, generating equations representative of the component voltage drops and polarities. As with the branch current method, we will denote a resistor's voltage drop as the product of the resistance (in ohms) and its respective mesh current (that quantity being unknown at this point). Where two currents mesh, we will write that term in the equation with resistor current being the *sum* of the two meshing currents.

Tracing the left loop of the circuit, starting from the upper-left corner and moving counter-clockwise (the choice of starting points and directions is ultimately irrelevant), counting polarity as if we had a voltmeter in hand, red lead on the point ahead and black lead on the point behind, we get this equation: $\mathbf{-28 + 2(I_1 + I_2) + 4I_1 = 0}$

Notice that the middle term of the equation uses the sum of mesh currents I_1 and I_2 as the current through resistor R_2. This is because mesh currents I_1 and I_2 are going the same direction through R_2, and thus complement each other. Distributing the coefficient of 2 to the I_1 and I_2terms, and then combining I_1 terms in the equation, we can simplify as such:

Original Form of equation: $\mathbf{-28 + 2(I_1 + I_2) + 4I_1 = 0}$

Distribute terms: $\mathbf{-28 + 2I_1 + 2I_2 + 4I_1 = 0}$

Combine like terms: $\mathbf{-28 + 6I_1 + 2I_2 = 0}$

At this time we have one equation with two unknowns. To be able to solve for two unknown mesh currents, we must have two equations. If we trace the other loop of the circuit, we can obtain another KVL equation and have enough data to solve for the two currents.

Starting at the upper-left hand corner of the right loop and tracing counter-clockwise:

$$-2(I_1 + I2) + 7 - 1I_2 = 0$$

Simplify the equation as before:

$$-2I_1 - 3I_2 + 7 = 0$$

Now, with two equations, we can use one of several methods to solve for the unknown currents I_1 and I_2:

$$-28 + 6I_1 + 2I_2 = 0$$

$$-2I_1 - 3I_2 + 7 = 0$$

Rearrange equations for easier solution:

$$6I_1 + 2I_2 = 28$$

$$-2I_1 - 3I_2 = -7$$

Solutions:

$$I_1 = 5\ A$$

$$I_2 = -1\ A$$

Knowing that these solutions are values for *mesh* currents, not *branch* currents, we must return to our diagram to see how they fit together to give currents through all components as shown in Figure 11.

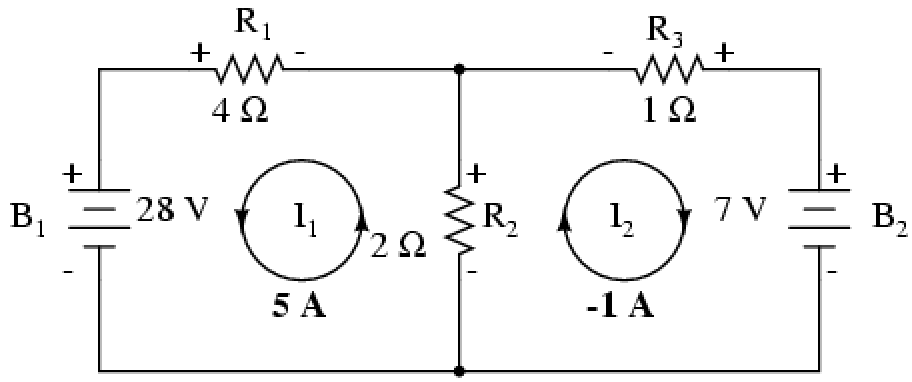


Figure 11: Original current directions for each mesh.

The solution of -1 amp for I_2 means that our initially assumed direction of current was incorrect. In actuality, I_2 is flowing in a counter-clockwise direction at a value of (positive) 1 amp as shown in Figure 12.

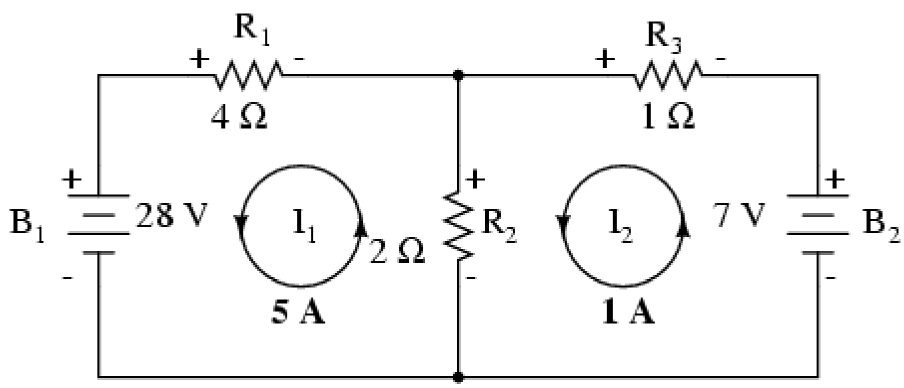


Figure 12: Corrected current directions for I_2.

This change of current direction from what was first assumed will alter the polarity of the voltage drops across R_2 and R_3 due to current I_2. From here, we can say that the current through R_1 is 5 amps, with the voltage drop across R_1 being the product of current and resistance (E=IR), 20 volts (positive on the left and negative on the right). Also, we can safely say that the current through R_3 is 1 amp, with a voltage drop of 1 volt (E=IR), positive on the left and negative on the right. However, what is happening at R_2?

Mesh current I_1 is going "up" through R_2, while mesh current I_2 is going "down" through R_2. To determine the actual current through R_2, we must see how mesh currents I_1 and I_2 interact (in this case they're in opposition), and algebraically add them to arrive at a final value. Since I_1 is going "up" at 5 amps, and I_2 is going "down" at 1 amp, the *real* current through R_2 must be a value of 4 amps, going "up" (Figure 13).

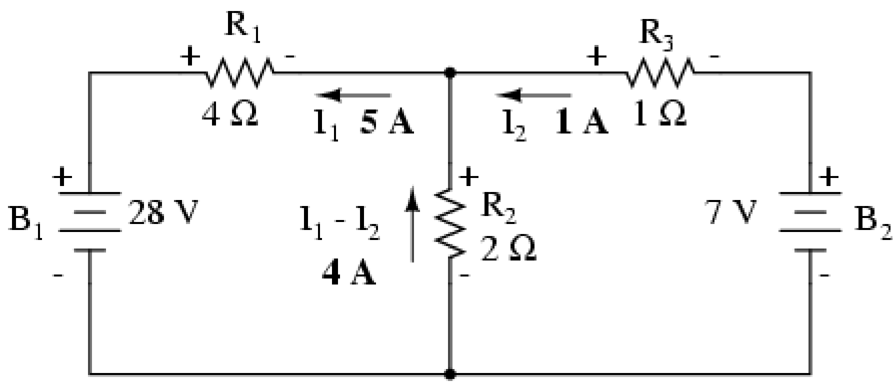


Figure 13: Direction of current flow.

A current of 4 amps through R_2's resistance of 2 Ω gives us a voltage drop of 8 volts (E=IR), positive on the top and negative on the bottom.

Steps to follow for the Mesh Current Method

- Draw mesh currents in loops of circuit, enough to account for all components.
- Label resistor voltage drop polarities based on assumed directions of mesh currents.
- Write KVL equations for each loop of the circuit, substituting the product IR for E in each resistor term of the equation. Where two mesh currents intersect through a component, express the current as the algebraic sum of those two mesh currents (i.e. I1 + I2) if the currents go in the same direction through that component. If not, express the current as the difference (i.e. I1 - I2).
- Solve for unknown mesh currents (simultaneous equations).
- If any solution is negative, then the assumed current direction is wrong!

- Algebraically add mesh currents to find current in components sharing multiple mesh currents.
- Solve for voltage drops across all resistors (E=IR).

Solving More Complex Circuits

The primary advantage of mesh current analysis is that it generally allows for the solution of a large network with fewer unknown values and fewer simultaneous equations. The previous example problem took three equations to solve using the branch current method and only two equations using the mesh current method. For more complex networks (Figure 14) the advantage afforded by mesh current analysis becomes evident.

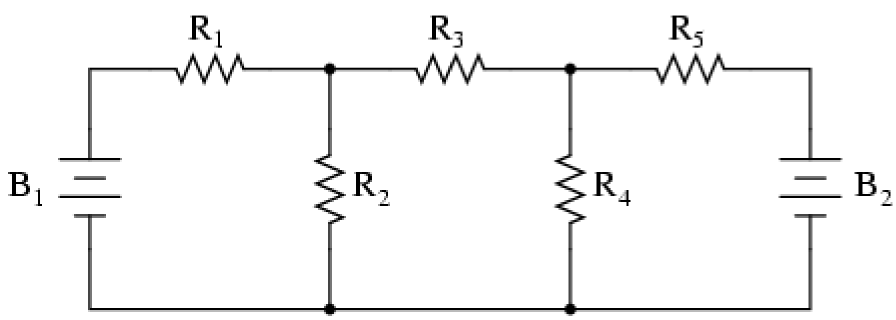
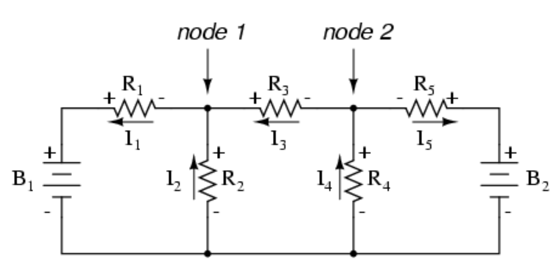


Figure 14: Complex series-parallel circuit with multiple batteries.

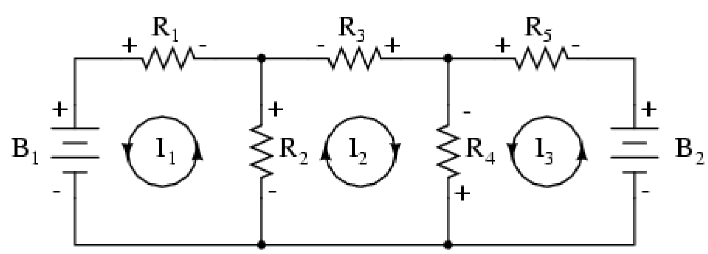
To solve this network using branch currents, five variables would need to be used to account for each unique current in the circuit (Figure 15).



$-I_1 + I_2 + I_3 = 0$	KCL at node 1
$-I_3 + I_4 - I_5 = 0$	KCL at node 2
$-E_{B1} + I_2R_2 + I_1R_1 = 0$	KVL in left loop
$-I_2R_2 + I_4R_4 + I_3R_3 = 0$	KVL in center loop
$-I_4R_4 + E_{B2} - I_5R_5 = 0$	KVL in right loop

Figure 15: Solving a complex series-parallel circuit using branch current method.

As shown in Figure 16, the mesh current method is a more effective choice, requiring only three unknowns and three equations to solve. Working with fewer equations is an advantage, especially when performing simultaneous equation solution by hand without a calculator.



KVL in left loop	
$-E_{B1} + R_2(I_1 + I_2) + I_1R_1 = 0$	
KVL in center loop	
$-R_2(I_2 + I_1) - R_4(I_2 + I_3) - I_2R_3 = 0$	
KVL in right loop	
$R_4(I_3 + I_2) + E_{B2} + I_3+R_5 = 0$	

Figure 16: Solving a complex series-parallel circuit using mesh current method.

The Wheatstone Bridge

Another type of circuit that lends itself well to mesh current method is the unbalanced Wheatstone bridge as shown in Figure 17.

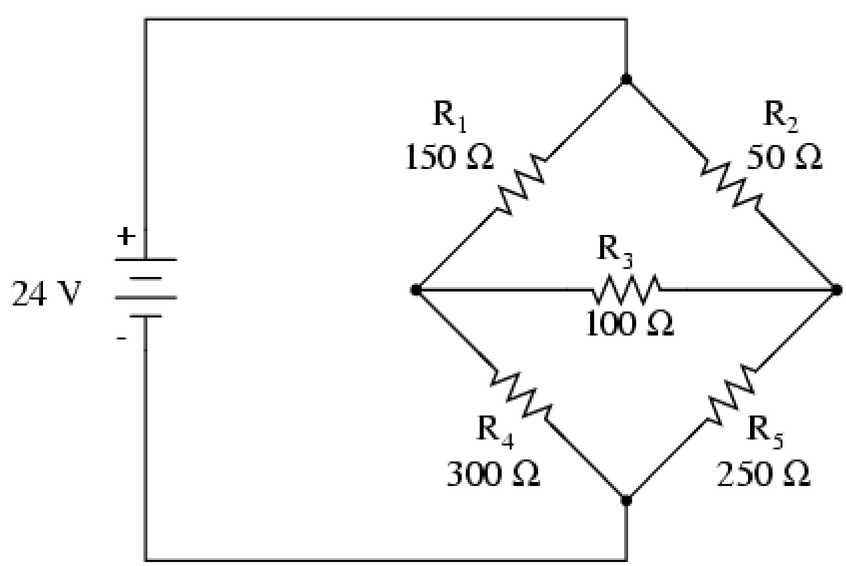


Figure 17: Unbalanced Wheatstone bridge circuit.

Because the ratios of R_1 to R_4 and R_2 to R_5 are unequal, there will be voltage across resistor R_3, and some amount of current through it. As discussed at the beginning of this module, this type of circuit is irreducible by normal series-parallel analysis, and may only be analyzed by some other method.

The branch current method would require a large set of simultaneous equations due to the multiple (six) current paths. The mesh current method, however, provides a less complicated means for solving this and other complicated circuits. The first step in the mesh current method is to draw the currents in each mesh section, or loop. The placement of the first two currents should be obvious, as shown below.

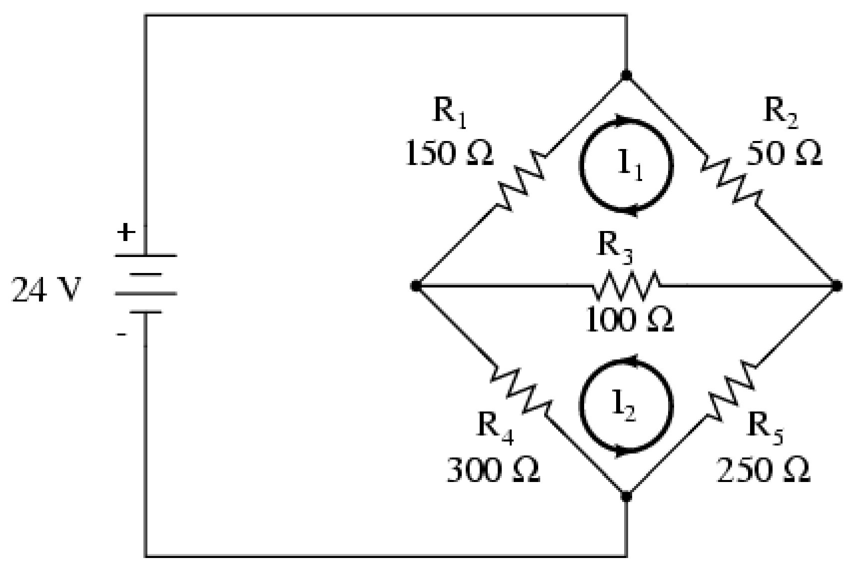


Figure 18: Drawing mesh currents in an unbalanced bridge.

As previously discussed, the initial directions of these mesh currents is arbitrary. However, two mesh currents are not enough in this circuit, because neither I_1 nor I_2 goes through the battery. Therefore, a third mesh current, I_3, needs to be added.

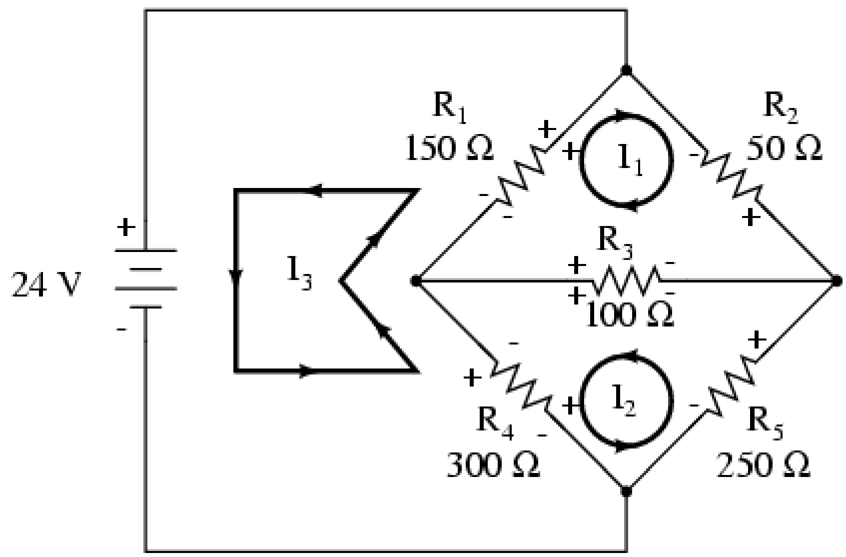


Figure 19: Mesh currents and voltage polarities.

In Figure 19, I_3 loops from the bottom side of the battery, through R_4, through R_1, and back to the top side of the battery. This is not the only path available for I_3, but it is the simplest. With the mesh currents identified, label the polarity of the voltage drops across each resistor, following each of the assumed currents' directions.

Notice something very important here: at resistor R_4, the polarities for the respective mesh currents do not agree. This is because those mesh currents (I_2 and I_3) are going through R_4 in different directions. While this does not preclude the use of the mesh current method for analysis, it does complicate it.

Generating a KVL equation for the top loop of the bridge, start at the top node and trace in a clockwise direction: $\mathbf{50I_1 + 100(I_1 + I_2) + 150(I_1 + I_3) = 0}$

Distribute terms: $\mathbf{50I_1 + 100I_1 + 100I_2 + 150I_1 + 150I_3 = 0}$

Combine like terms: $\mathbf{300I_1 + 100I_2 + 150I_3 = 0}$

In this equation, the common directions of currents are represented by their *sums* through common resistors. For example, resistor R_3, with a value of 100 Ω, has its voltage drop represented in the above KVL equation by the expression $100(I_1 + I_2)$, since both currents I_1 and I_2 go through $R_3$from right to left. The same exists for resistor R_1, with its voltage drop expression shown as $150(I_1 + I_3)$, since both I_1 and I_3 go from bottom to top through that resistor, and thus work *together* to generate its voltage drop.

Starting at the right-hand node, trace counter-clockwise:

$$\mathbf{100(I_1 + I_2) + 300(I_2 - I_3) + 250I_2 = 0}$$

Distribute terms: $\mathbf{100I_1 + 100I_2 + 300I_2 - 300I_3 + 250I_2 = 0}$

Combine like terms: $\mathbf{100I_1 + 650I_2 - 300I_3 = 0}$

Note how the second term in the equation's original form has resistor R_4's value of 300 Ω multiplied by the *difference* between I_2 and I_3 ($I_2 - I_3$). This is how we represent the combined effect of two mesh currents going in opposite directions through the same component. Choosing the appropriate mathematical signs is very important here: $300(I_2 - I_3)$ does not mean the same thing as $300(I_3 - I_2)$.

The third equation must include the battery's voltage, which up to this point does not appear in either of the previous KVL equations.

Trace a loop starting from the battery's bottom (negative) terminal, stepping clockwise:

$$\mathbf{24 - 150(I_3 + I_1) - 300(I_3 - I_2) = 0}$$

Distribute terms: $\mathbf{24 - 150I_3 - 150I_1 - 300I_3 + 300I_2 = 0}$

Combine terms: $\mathbf{-150I_1 + 300I_2 - 450I_3 = -24}$

We now have the equations necessary to solve this circuit:

Top loop of bridge	$\mathbf{300I_1 + 100I_2 + 150I_3 = 0}$
Bottom loop of bridge	$\mathbf{100I_1 + 650I_2 - 300I_3 = 0}$
Bottom loop, including battery	$\mathbf{-150I_1 + 300I_2 - 450I_3 = -2}$

Solution: I1 = -93.793 mA
I2 = 77.241 mA
I3 = 136.092 mA

Node voltage method

The node voltage method of analysis solves for unknown voltages at circuit nodes in terms of a system of KCL equations. This analysis involves replacing voltage sources with equivalent current sources. Also, resistor values in ohms are replaced by equivalent conductances in siemens (S), G = 1/R.

Start with a circuit having conventional voltage sources. A common node E_0 is chosen as a reference point. The node voltages E_1 and E_2 are calculated with respect to this point.

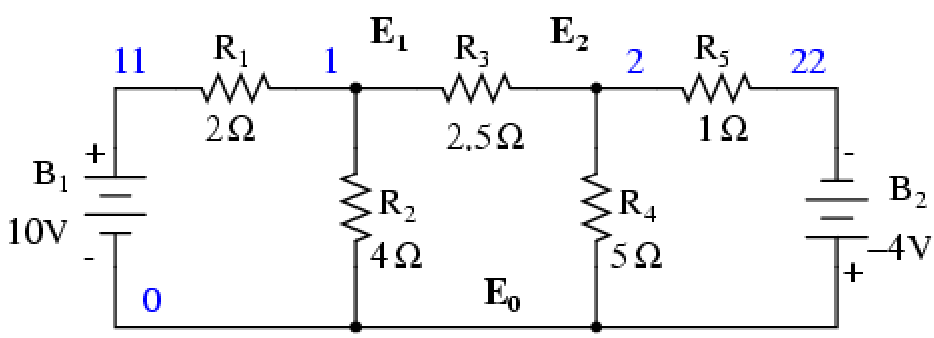


Figure 20: Node voltage method for solving circuit unknowns

A voltage source in series with a resistance must be replaced by an equivalent current source in parallel with the resistance. Write KCL equations for each node. The right hand side of the equation is the value of the current source feeding the node. The symbol for a current source is a circle with an arrow pointing downwards. Please note that this is a convention and does not indicate current direction.

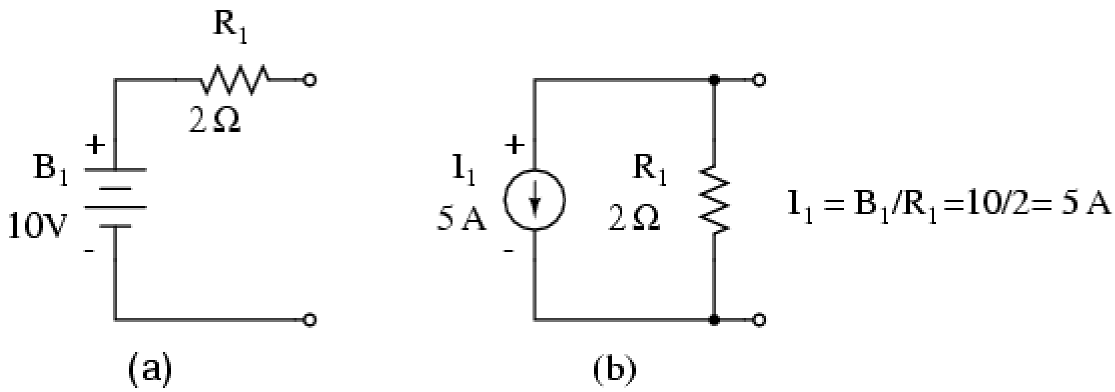


Figure 21: Replacing voltage source with source current.

Replacing voltage sources and associated series resistors with equivalent current sources and parallel resistors yields the modified circuit. Substitute resistor conductances in siemens for resistance in ohms.

Calculate current sources for B_1 and B_2:

$I_1 = E_1/R_1 = 10/2 = 5\text{ A}$

$I_2 = E_2/R_5 = 4/1 = 4\text{ A}$

Calculate conductance of each resistor:

$G1 = 1/R1 = 1/2\ \Omega = 0.5\text{ S}$

$G2 = 1/R2 = 1/4\ \Omega = 0.25\text{ S}$

$G3 = 1/R3 = 1/2.5\ \Omega = 0.4\text{ S}$

$G4 = 1/R4 = 1/5\ \Omega = 0.2\text{ S}$

$G5 = 1/R5 = 1/1\ \Omega = 1.0\text{ S}$

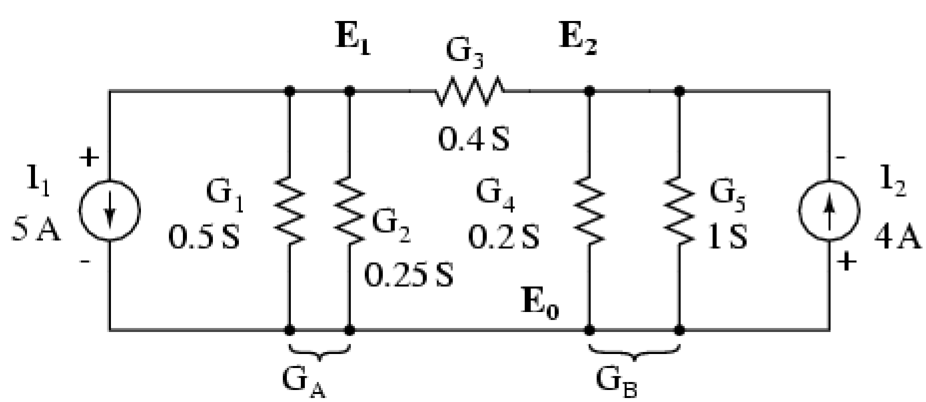


Figure 22: Circuit with current sources and conductances

The Parallel conductances (resistors) may be combined by addition of the conductances. Though, we will not redraw the circuit. The circuit is ready for application of the node voltage method.

$$\mathbf{G_A = G_1 + G_2 = 0.5\ S + 0.25\ S = 0.75\ S}$$
$$\mathbf{G_B = G_4 + G_5 = 0.2\ S + 1\ S = 1.2\ S}$$

Deriving a general node voltage method, write a pair of KCL equations in terms of unknown node voltages V_1 and V_2 this one time. This is done to illustrate a pattern for writing equations by inspection.

$$\mathbf{G_AE_1 + G_3\ (E_1 - E_2) = I_1 \qquad (1)}$$
$$\mathbf{G_BE_2 - G_3\ (E_1 - E_2) = I_2 \qquad (2)}$$
$$\mathbf{(G_A + G_3)\ E_1 - G_3E_2 = I_1 \qquad (1)}$$
$$\mathbf{-G_3E_1 + (G_B + G_3)\ E2 = I_2 \qquad (2)}$$

The coefficients of the last pair of equations above have been rearranged to show a pattern. The sum of conductances connected to the first node is the positive coefficient of the first voltage in equation (1). The sum of conductances connected to the second node is the positive coefficient of the second voltage in equation (2). The other coefficients are negative, representing conductances between nodes. For both equations, the right hand side is equal to the respective current source connected to the node. This pattern allows us to quickly write the equations by inspection. This leads to a set of rules for the node voltage method of analysis.

Node voltage rules

1. Convert voltage sources in series with a resistor to an equivalent current source with the resistor in parallel.
2. Change resistor values to conductances.
3. Select a reference node(E_0)

4. Assign unknown voltages (E_1) (E_2) ... (E_N) to remaining nodes.

5. Write a KCL equation for each node 1, 2, ... N. The positive coefficient of the first voltage in the first equation is the sum of conductances connected to the node. The coefficient for the second voltage in the second equation is the sum of conductances connected to that node. Repeat for coefficient of third voltage, third equation, and other equations. These coefficients fall on a diagonal.

6. All other coefficients for all equations are negative, representing conductances between nodes. The first equation, second coefficient is the conductance from node 1 to node 2; the third coefficient is the conductance from node 1 to node 3. Fill in negative coefficients for other equations.

7. The right hand side of the equations is the current source connected to the respective nodes.

8. Solve system of equations for unknown node voltages.

Introduction to Network Theorems

A **theorem** is a relatively simple rule used to solve a problem and is derived from a more intensive analysis using fundamental rules of mathematics. In electric network analysis, the fundamental rules are Ohm's law and Kirchhoff's laws. While these laws may be applied to analyze nearly any circuit configuration, there are some shortcut methods of analysis to make the math more manageable.

As with any theorem of algebra, these network theorems are derived from fundamental rules. In this module, the formal proofs of these theorems will not be presented, however, if desired, you can empirically test them by setting up example circuits and calculating values using the simultaneous equation method versus the theorems, to see if the answers coincide.

Millman's Theorem

In **Millman's theorem**, the circuit is re-drawn as a parallel network of branches, each branch containing a resistor or series battery-resistor combination. Millman's theorem is applicable only to those circuits which can be re-drawn accordingly. Here again is the example circuit used for the previous methods of analysis, but re-drawn for the sake of applying Millman's theorem.

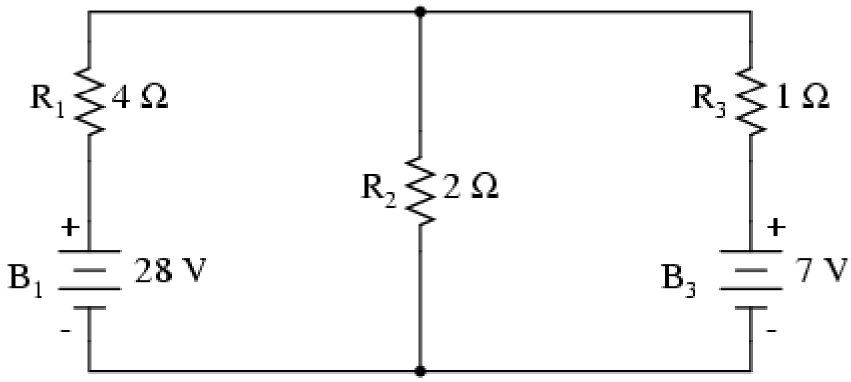


Figure 23: Series-parallel circuit redrawn for Millman's theorem.

By considering the supply voltage within each branch and the resistance within each branch, Millman's theorem will tell us the voltage across all branches. Please note the battery in the rightmost branch is labeled as "B_3" to clearly denote it as being in the third branch, even though there is no "B_2" in the circuit.

Millman's theorem is nothing more than a long equation, applied to any circuit drawn as a set of parallel-connected branches with each branch having its own voltage source and series resistance.

Millman's Theorem Equation

$$\frac{\frac{E_{B1}}{R_1} + \frac{E_{B2}}{R_2} + \frac{E_{B3}}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \text{Voltage across all branches}$$

Figure 24: Millman's theorem equation.

Substituting actual voltage and resistance figures from our example circuit for the variable terms of this equation, we get the following expression:

$$\frac{\dfrac{28\ \text{V}}{4\ \Omega} + \dfrac{0\ \text{V}}{2\ \Omega} + \dfrac{7\ \text{V}}{1\ \Omega}}{\dfrac{1}{4\ \Omega} + \dfrac{1}{2\ \Omega} + \dfrac{1}{1\ \Omega}} = 8\ \text{V}$$

Figure 25: Millman's theorem equation for sample circuit.

The final answer of 8 volts is the voltage seen across all parallel branches, like this:

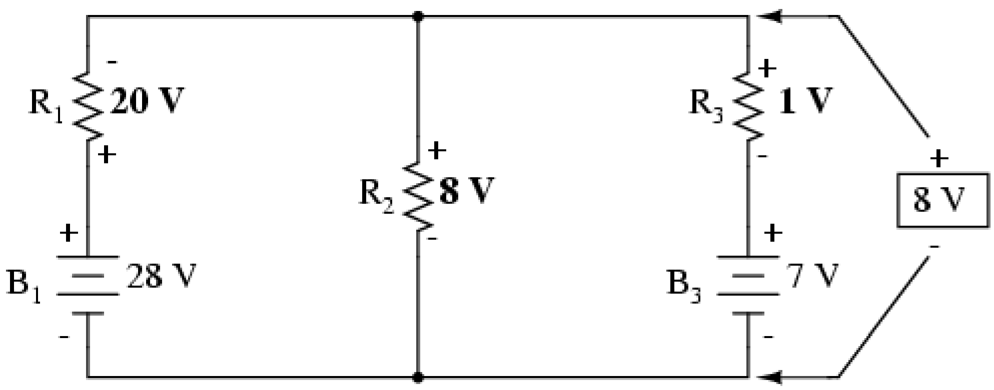


Figure 26: Circuit voltages redrawn according to Millman's theorem.

The polarities of all voltages in Millman's theorem are referenced to the same point. In the example circuit above, the bottom wire of the parallel circuit is the reference point; therefore, the voltages within each branch (28 for the R_1 branch, 0 for the R_2 branch, and 7 for the R_3 branch) were inserted into the equation as positive numbers. Likewise, when the answer came out to 8 volts (positive), this meant that the top wire of the circuit was positive with respect to the bottom wire (the original point of reference). If both batteries had been connected backwards (negative ends up and positive ends down), the voltage for branch 1 would have been entered into the equation as a -28 volts, the voltage for branch 3 as -7 volts, and the resulting answer of -8 volts would have indicated that the top wire was negative with respect to the bottom wire (the initial point of reference).

To solve for resistor voltage drops, the Millman voltage (across the parallel network) must be compared against the voltage source within each branch, using the principle of voltages adding in series to determine the magnitude and polarity of voltage across each resistor:

$$E_{R1} = 8\text{ V} - 28\text{ V} = -20\text{ V} \quad \text{(negative on top)}$$

$$E_{R2} = 8\text{ V} - 0\text{ V} = 8\text{ V} \quad \text{(positive on top)}$$

$$E_{R3} = 8\text{ V} - 7\text{ V} = 1\text{ V} \quad \text{(positive on top)}$$

Figure 27: Solving for resistor voltage drops.

To solve for branch currents, each resistor voltage drop can be divided by its respective resistance (I=E/R):

$$I_{R1} = \frac{20\text{ V}}{4\ \Omega} = 5\text{ A}$$

$$I_{R2} = \frac{8\text{ V}}{2\ \Omega} = 4\text{ A}$$

$$I_{R3} = \frac{1\text{ V}}{1\ \Omega} = 1\text{ A}$$

Figure 28: Solving for branch currents.

The direction of current through each resistor is determined by the polarity across each resistor, *not* by the polarity across each battery, as current can be forced backwards through a battery, as is the case with B_3 in the example circuit. This is important to keep in mind, since Millman's theorem doesn't provide as direct an indication of “wrong” current direction as does the branch current or mesh current methods. You must pay close attention to the polarities of resistor voltage drops as given by Kirchhoff's voltage law, determining direction of currents from that.

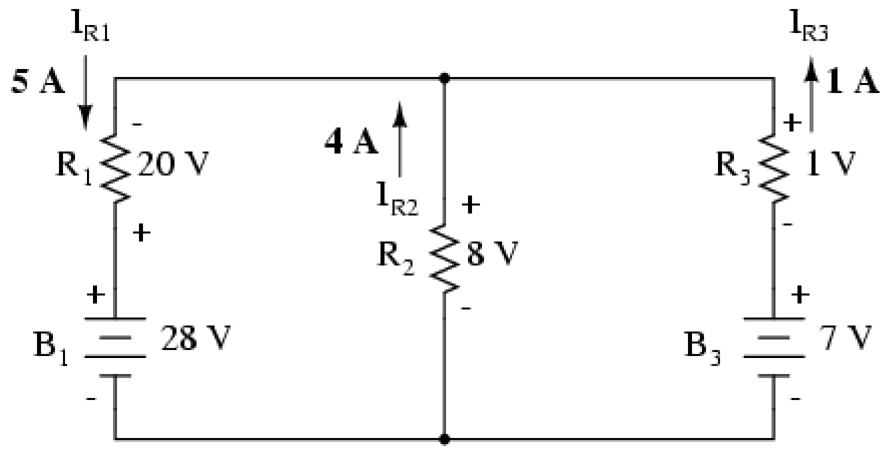


Figure 29: Polarities and direction of current flow.

Millman's theorem is very convenient for determining the voltage across a set of parallel branches, where there are enough voltage sources present to preclude solution via regular series-parallel reduction method. It also is easy in the sense that it doesn't require the use of simultaneous equations. However, it is limited in that it only applied to circuits which can be re-drawn to fit this form. It cannot be used, for example, to solve an unbalanced bridge circuit. Even in cases where Millman's theorem can be applied, the solution of individual resistor voltage drops can be daunting with Millman's theorem equation only providing a single figure for branch voltage.

Each network analysis method has its own advantages and disadvantages. Each method is a tool, and no tool is perfect for all jobs. The skilled technician, however, carries these methods in his or her mind like a mechanic carries a set of tools in his or her tool box. The more tools you have equipped yourself with, the more prepared you will be for any eventuality.

Superposition Theorem

Superposition theorem takes a complex subject and simplifies it in a way that makes perfect sense. A theorem like Millman's certainly works well, but it is not quite obvious *why* it works so well. Superposition, on the other hand, is obvious.

The strategy used in the superposition theorem is to eliminate all but one source of power within a network at a time, using series-parallel analysis to determine voltage drops (and/or currents) within the modified network for each power source separately. Then, once voltage drops and/or currents have been determined for each power source working separately, the values are all "superimposed" on top of each other (added algebraically) to find the actual voltage drops or currents with all sources active.

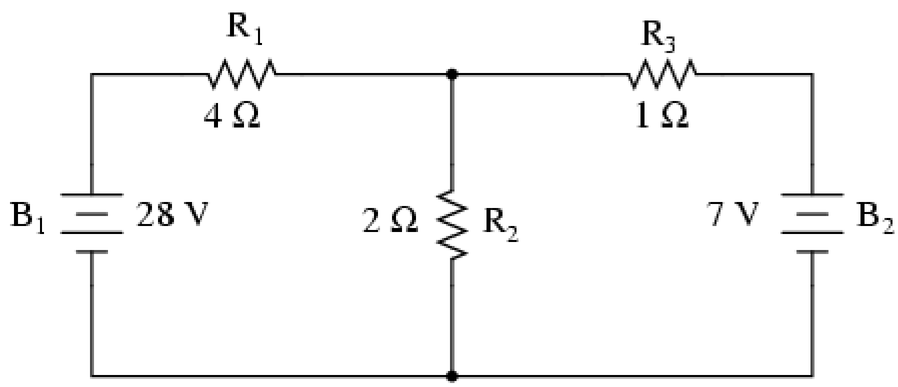


Figure 30: Sample circuit with multiple voltage sources.

Because there are two sources of power in this circuit, two sets of values for voltage drops and/or currents must be calculated, one for the circuit with only the 28 volt battery in effect, and one for the circuit with only the 7 volt battery in effect.

When re-drawing the circuit for series-parallel analysis with one source, all other voltage sources are replaced by shorts (wires) and all current sources with opens (breaks). Since there are only voltage sources (batteries) in the example circuit, replace each inactive source during analysis with a wire.

Analyzing the circuit with only the 28-volt battery, we obtain the following values for voltage and current:

	R_1	R_2	R_3	$R_2//R_3$	$R_1 + R_2//R_3$ Total	
E	24	4	4	4	28	Volts
I	6	2	4	6	6	Amps
R	4	2	1	0.667	4.667	Ohms

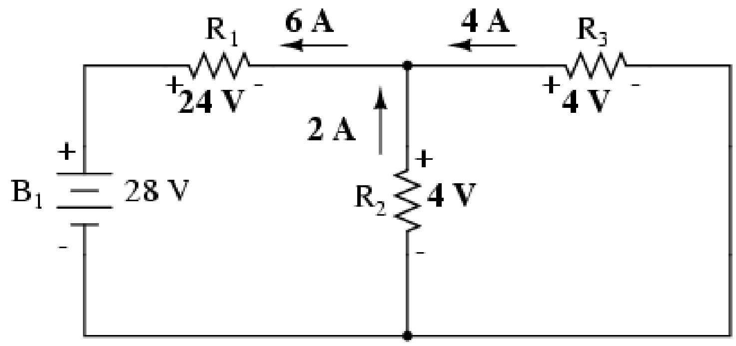


Figure 31: Analysis of circuit with 28 V source.

Analyzing the circuit with only the 7-volt battery, we obtain another set of values for voltage and current:

	R_1	R_2	R_3	$R_1//R_2$	$R_3 + R_1//R_2$ Total	
E	4	4	3	4	7	Volts
I	1	2	3	3	3	Amps
R	4	2	1	1.333	2.333	Ohms

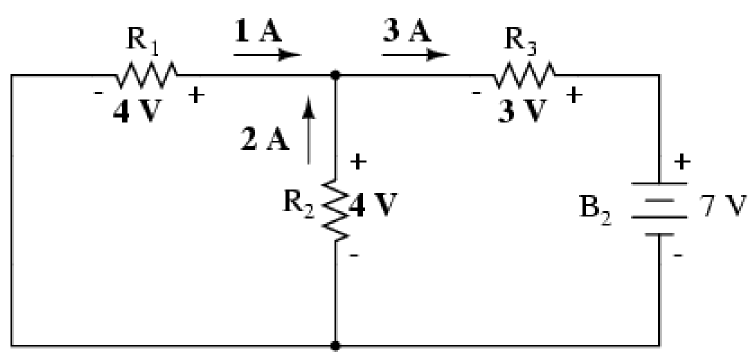


Figure 32: Analysis of circuit with 7 V source.

When superimposing these values of voltage and current, to be very careful to consider polarity (voltage drop) and direction (electron flow), as the values have to be added *algebraically*.

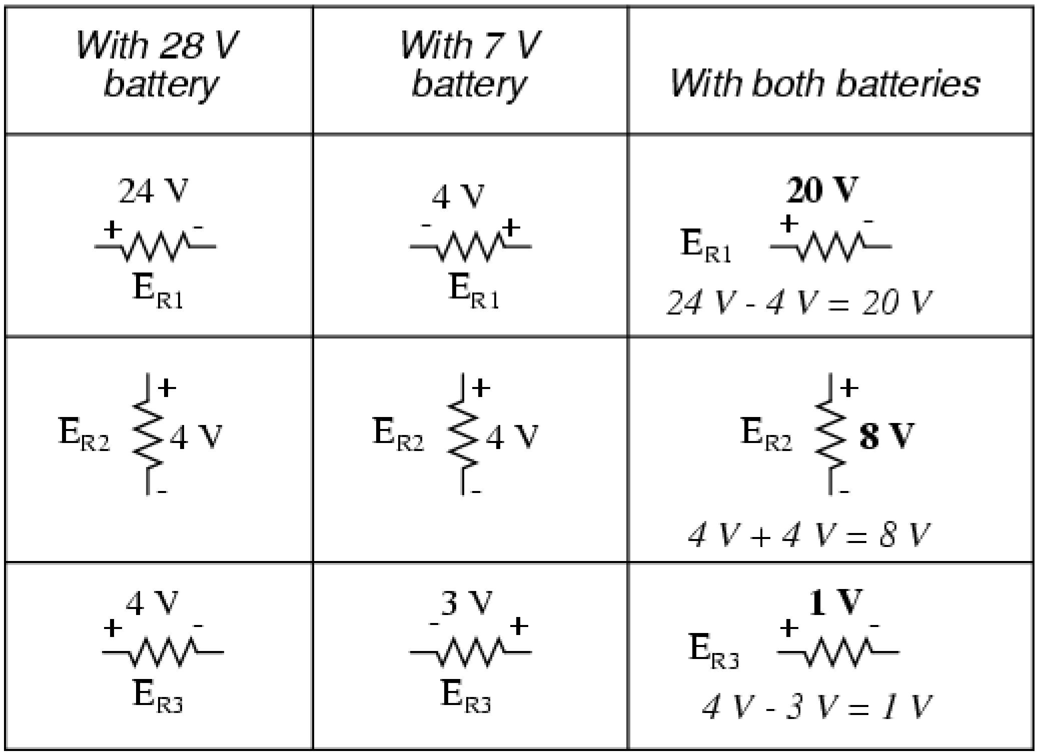


Figure 33: Summary of voltages results.

Applying these superimposed voltage figures to the circuit; the result looks something like this:

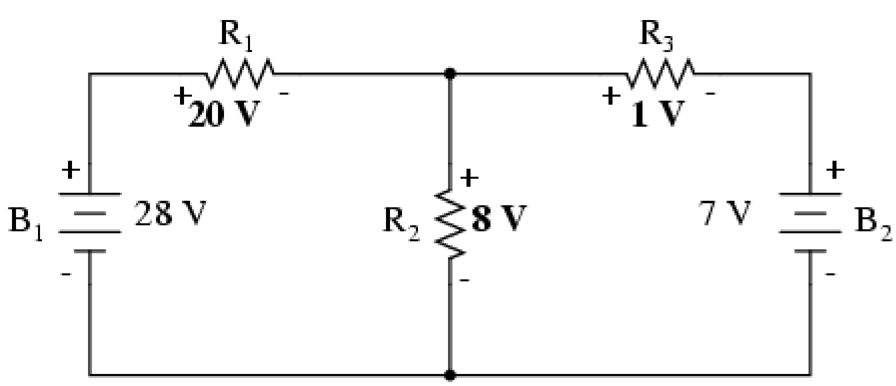


Figure 34: Circuit redrawn with superimposed voltages.

Currents add up algebraically as well, and can either be superimposed as done with the resistor voltage drops, or simply calculated from the final voltage drops and respective resistances (I=E/R). Either way, the answers will be the same.

Here is the superposition method as applied to current:

With 28 V battery	*With 7 V battery*	*With both batteries*
I_{R1} 6 A (←)	I_{R1} 1 A (→)	I_{R1} **5 A** (←) *6 A - 1 A = 5 A*
I_{R2} 2 A (↑)	I_{R2} 2 A (↑)	I_{R2} **4 A** (↑) *2 A + 2 A = 4 A*
I_{R3} 4 A (←)	I_{R3} 3 A (→)	I_{R3} **1 A** (←) *4 A - 3 A = 1 A*

Figure 35: Summary of current results

Once again applying these superimposed figures to the circuit:

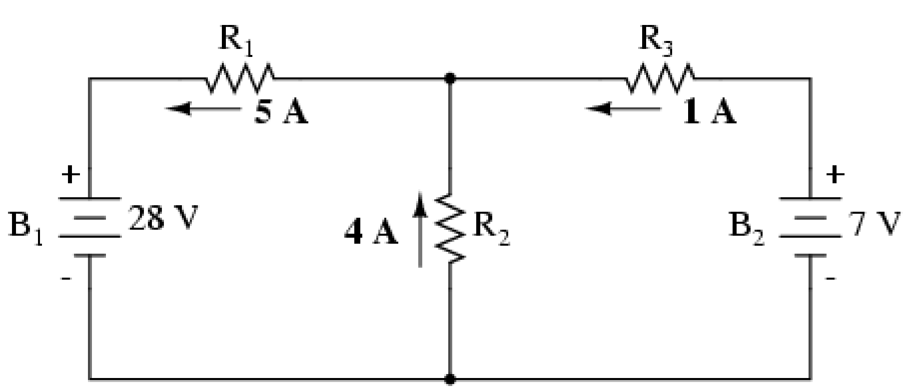


Figure 36: Circuit redrawn with superimposed currents.

It must be noted, though, that the superposition theorem works only for circuits that are reducible to series-parallel combinations for each of the power sources at a time and works only where the underlying equations are linear. The requisite of linearity means that superposition theorem is only applicable for determining voltage and current, not power. Power dissipations are nonlinear functions and do not algebraically add to an accurate total when only one source is considered at a time. The need for linearity also means this theorem cannot be applied in circuits where the resistance of a component changes with voltage or current. Hence, networks containing components like lamps (incandescent or gas-discharge) or varistors could not be analyzed.

Another prerequisite for superposition theorem is that all components must be **bilateral**, meaning that they behave the same with electrons flowing either direction through them. Resistors have no polarity-specific behavior, and so the circuits studied so far all meet this criterion.

The superposition theorem finds use in the study of alternating current (AC) circuits, and semiconductor (amplifier) circuits, where sometimes AC is often mixed (superimposed) with DC. Because AC voltage and current equations (Ohm's law) are linear just like DC, we can use superposition to analyze the circuit with just the DC power source, and then just the AC power source, combining the results to tell what will happen with both AC and DC sources in effect. For now, though, Superposition will suffice as a break from having to do simultaneous equations to analyze a circuit.

Thevenin's Theorem

Thevenin's theorem states that it is possible to simplify any linear circuit, no matter how complex, to an equivalent circuit with just a single voltage source and series resistance connected to a load. The qualification of linearity is identical to that found in the superposition theorem, where all the underlying equations must be linear (no exponents or roots). If we're dealing with **passive components** (such as resistors, and later, inductors and capacitors), this is true. However, there are some components (especially certain gas-discharge and semiconductor components) which are nonlinear: that is, their opposition to current *changes* with voltage. As such, we would call circuits containing these types of components, *nonlinear circuits*.

Thevenin's theorem is especially useful in analyzing power systems and other circuits where one particular resistor in the circuit (called the “load” resistor) is subject to change, and re-calculation of the circuit is necessary with each trial value of load resistance, to determine voltage across it and current through it. Take another look at our example circuit:

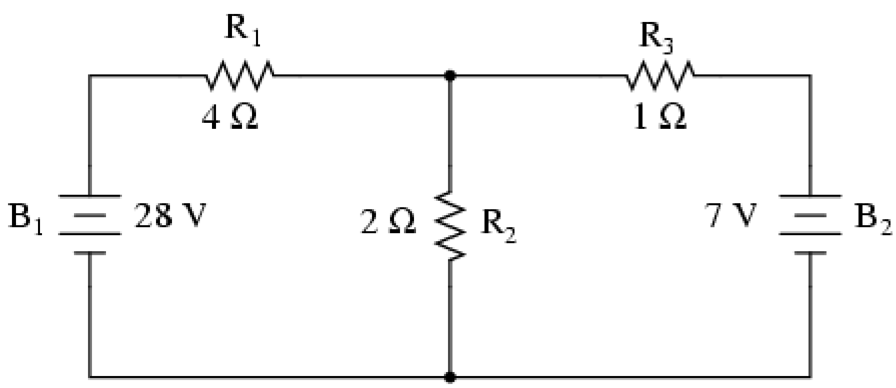


Figure 37: Sample series-parallel circuit with multiple voltage sources.

Suppose that we decide to designate R_2 as the “load” resistor in this circuit. We already have four methods of analysis at our disposal (branch current, mesh current, Millman's theorem, and superposition theorem) to use in determining voltage across R_2 and current through R_2, but each of these methods is time-consuming. Imagine repeating any of these methods repeatedly to find what would happen if the load resistance changed.

Thevenin's theorem makes this easy by temporarily removing the load resistance from the original circuit and reducing what's left to an equivalent circuit composed of a single voltage source and series resistance. The load resistance can then be re-connected to this Thevenin equivalent circuit and calculations carried out as if the whole network were nothing but a simple series circuit:

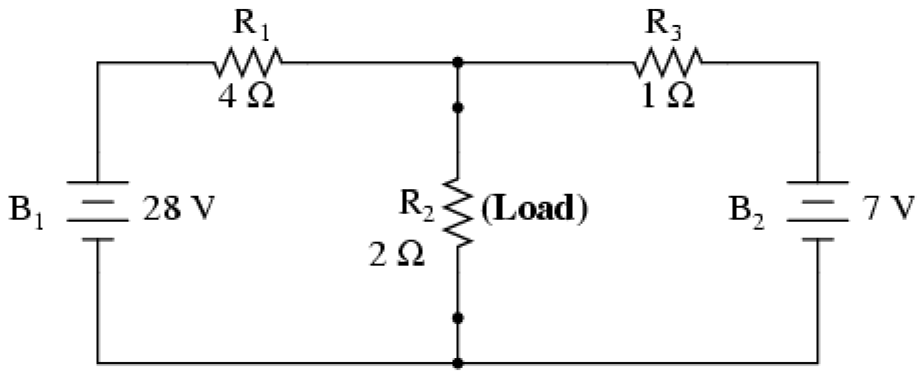


Figure 38: Sample circuit redrawn for Thevenin analysis.

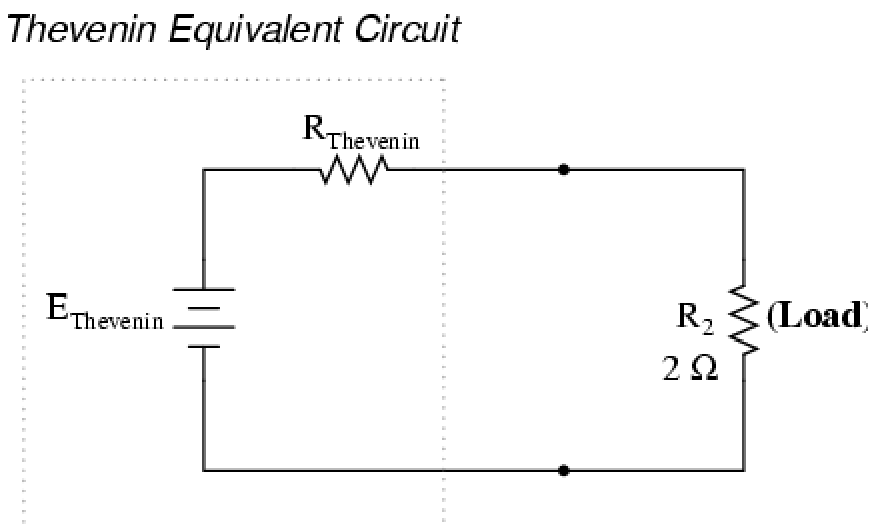


Figure 39: Thevenin equivalent of sample circuit.

The Thevenin equivalent circuit is the electrical equivalent of B_1, R_1, R_3, and B_2 as seen from the two points where our load resistor (R_2) connects.

The Thevenin equivalent circuit, if correctly derived, will behave exactly the same as the original circuit formed by B_1, R_1, R_3, and B_2. In other words, the load resistor (R_2) voltage and current should be exactly the same for the same value of load resistance in the two circuits. The load resistor R_2 cannot tell the difference between the original network of B_1, R_1, R_3, and B_2, and the Thevenin equivalent circuit of $E_{Thevenin}$, and $R_{Thevenin}$, provided that the values for $E_{Thevenin}$ and $R_{Thevenin}$ have been calculated correctly.

The advantage in performing the Thevenin conversion to the simpler circuit, of course, is that it makes load voltage and load current so much easier to solve than in the original network. Calculating the equivalent Thevenin source voltage and series resistance is actually quite easy. First, the chosen load resistor is removed from the original circuit, replaced with a break (open circuit):

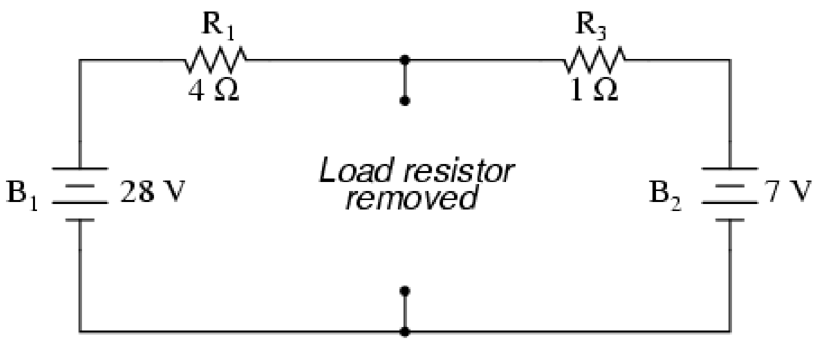


Figure 40: Step 1 – Open the load resistor.

Next, determine the voltage between the two points left by the removal of the load resistor In this case, the original circuit with the load resistor removed is nothing more than a simple series circuit with opposing batteries, and so we can determine the voltage across the open load terminals by applying the rules of series circuits, Ohm's law, and Kirchhoff's voltage law:

	R_1	R_3	Total	
E	16.8	4.2	21	Volts
I	4.2	4.2	4.2	Amps
R	4	1	5	Ohms

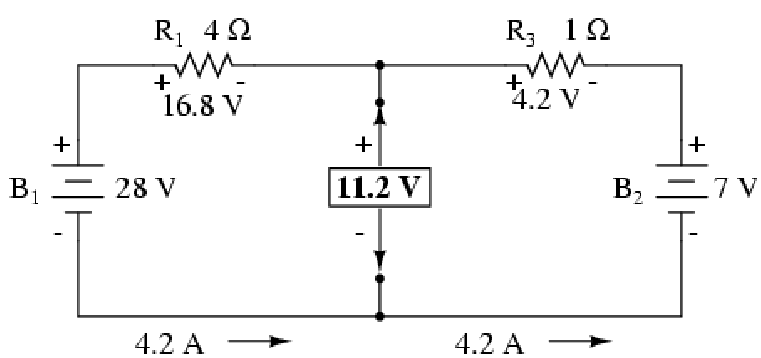


Figure 41: Calculate voltage drop of opened load.

The voltage between the two load connection points can be figured from the one of the battery's voltage and one of the resistor's voltage drops, and comes out to 11.2 volts. This is our Thevenin voltage ($E_{Thevenin}$) in the equivalent circuit:

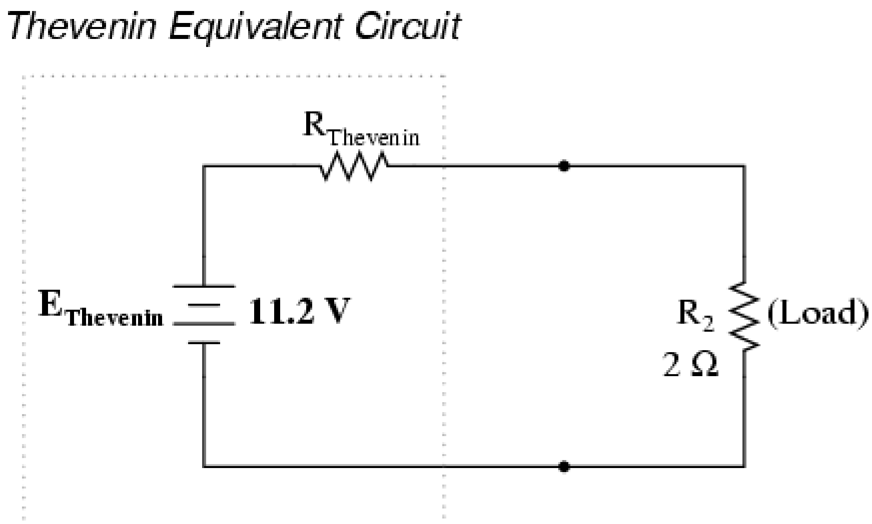


Figure 42: Thevenized voltage

To find the Thevenin series resistance for our equivalent circuit, we need to take the original circuit, with the load resistor still removed, and remove the power sources in the same style as we did with the superposition theorem (voltage sources replaced with wires and current sources replaced with breaks). Finally figure the resistance from one load terminal to the other:

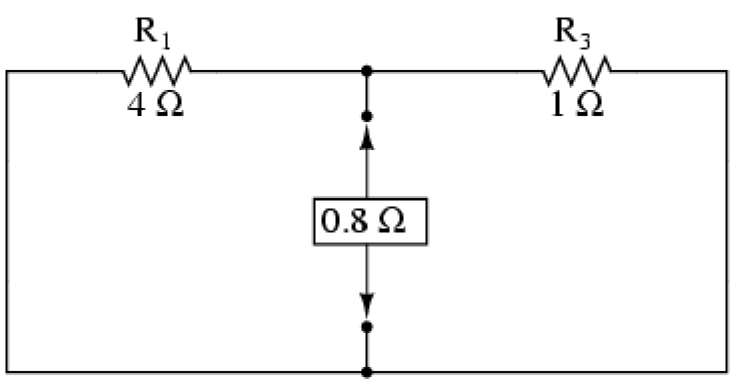


Figure 43: Step 2 – find Thevenin resistance by shorting the voltages.

With the removal of the two batteries, the total resistance measured at this location is equal to R_1 and R_3 in parallel: 0.8 Ω. This is our Thevenin resistance ($R_{Thevenin}$) for the equivalent circuit:

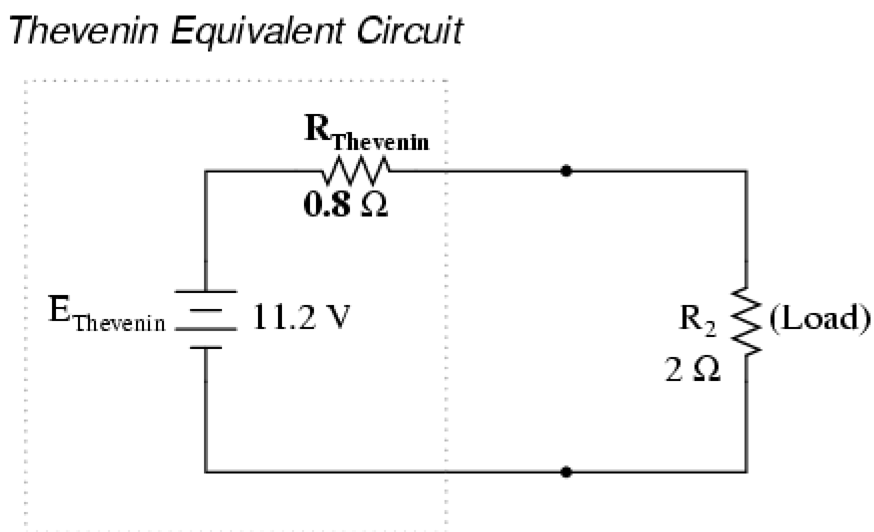


Figure 44: Step 3 Draw the Thevenin circuit.

With the load resistor (2 Ω) attached between the connection points, determine the voltage across it and current through it as though the whole network were nothing more than a simple series circuit:

	$R_{Thevenin}$	R_{Load}	Total	
E	3.2	8	11.2	Volts
I	4	4	4	Amps
R	0.8	2	2.8	Ohms

Figure 45: Step 4 – Analyze voltage and current for R_{Load}

Notice that the voltage and current figures for R_2 (8 volts, 4 amps) are identical to those found using other methods of analysis. Also notice that the voltage and current figures for the Thevenin series resistance and the Thevenin source (*total*) do not apply to any component in the original, complex circuit. Thevenin's theorem is only useful for determining what happens to a *single* resistor in a network, the load.

The advantage, of course, is that it can be quickly determined what would happen to that single resistor if it were to change.

Steps to follow for Thevenin's Theorem

1. Find the Thevenin source voltage by removing the load resistor from the original circuit and calculating voltage across the open connection points where the load resistor used to be.

2. Find the Thevenin resistance by removing all power sources in the original circuit (voltage sources shorted and current sources open) and calculating total resistance between the open connection points.

3. Draw the Thevenin equivalent circuit, with the Thevenin voltage source in series with the Thevenin resistance. The load resistor re-attaches between the two open points of the equivalent circuit.

4. Analyze voltage and current for the load resistor following the rules for series circuits.

Norton's Theorem

Norton's Theorem states that it is possible to simplify any linear circuit, no matter how complex, to an equivalent circuit with just a single current source and parallel resistance connected to a load. Just as with Thevenin's theorem all underlying equations must be linear.

Before Norton conversion, the original example circuit looks like this:

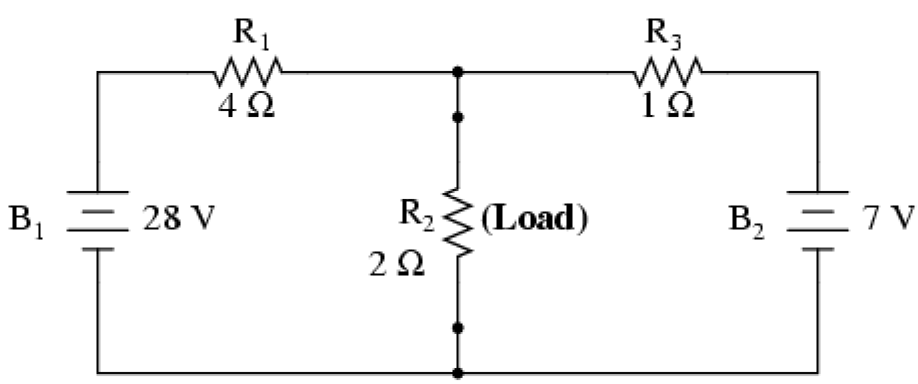


Figure 46: Sample circuit before Norton conversion.

After Norton conversion, the equivalent circuit looks like this:

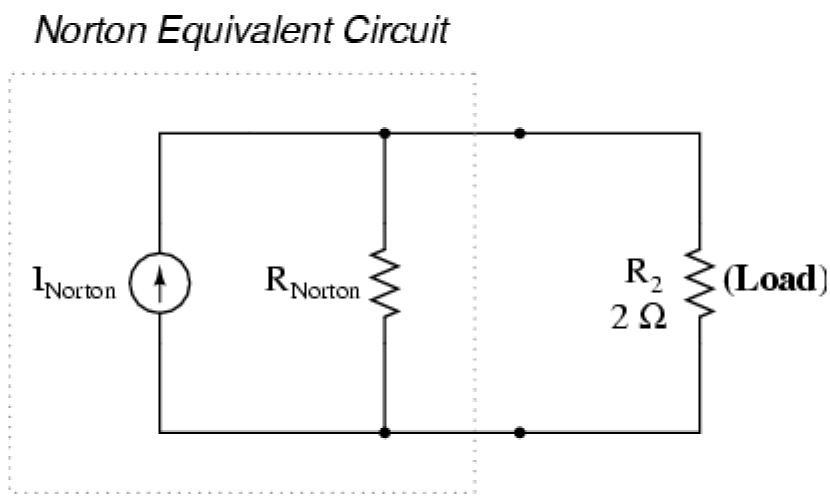


Figure 47: Sample circuit after Norton conversion.

Remember that a **current source** is a component whose job is to provide a constant amount of current, outputting as much or as little voltage necessary to maintain that constant current.

As with Thevenin's theorem, everything in the original circuit except the load resistance has been reduced to an equivalent circuit that is simpler to analyze. Also similar to Thevenin's theorem are the steps used in Norton's theorem to calculate the Norton source current (I_{Norton}) and Norton resistance (R_{Norton}).

As before, the first step is to identify the load resistance and remove it from the original circuit:

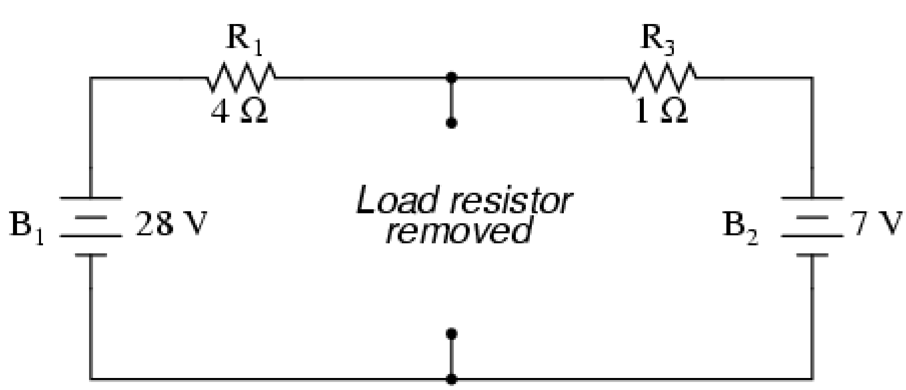


Figure 48: Step 1A – Identify and remove the load resistance.

Then, to find the Norton current (for the current source in the Norton equivalent circuit), place a direct wire (short) connection between the load points, and determine the resultant current. Note that this step is exactly opposite the respective step in Thevenin's theorem, where we replaced the load resistor with a break (open circuit):

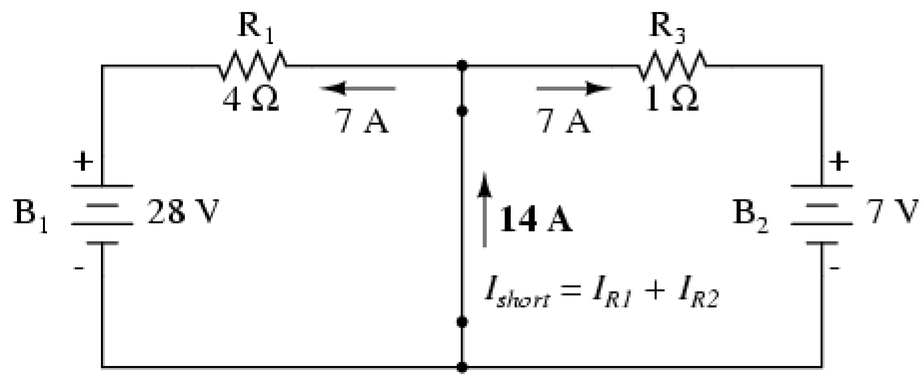


Figure 49: Step 1B – Replace the connection with a wire (Short).

With zero voltage dropped between the load resistor connection points, the current through R_1 is strictly a function of B_1's voltage and R_1's resistance: 7 amps (I=E/R). Likewise, the current through R_3 is now strictly a function of B_2's voltage and R_3's resistance: 7 amps (I=E/R). The total current through the short between the load connection points is the sum of these two currents: 7 amps + 7 amps = 14 amps. This figure of 14 amps becomes the Norton source current (I_{Norton}) in our equivalent circuit:

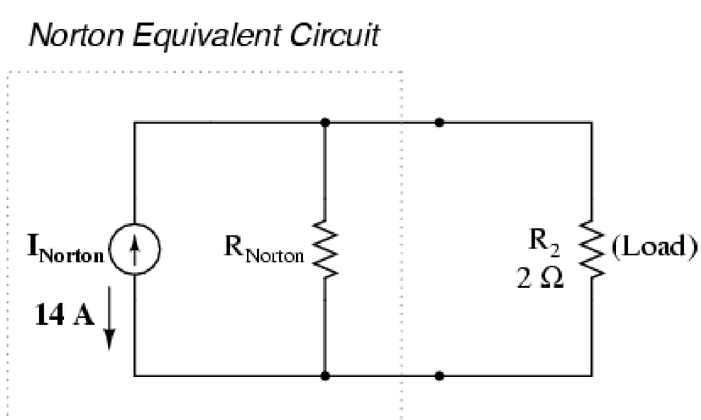


Figure 50: Norton equivalent circuit.

Remember, the symbol for a current source is standardized and does not reflect the direction of current flow.

To calculate the Norton resistance (R_{Norton}), follow the exact same process as for calculating Thevenin resistance ($R_{Thevenin}$). Starting with the original circuit, with the load resistor still removed, and remove the power sources in the same style as with the Superposition theorem. Finally figure total resistance from one load connection point to the other:

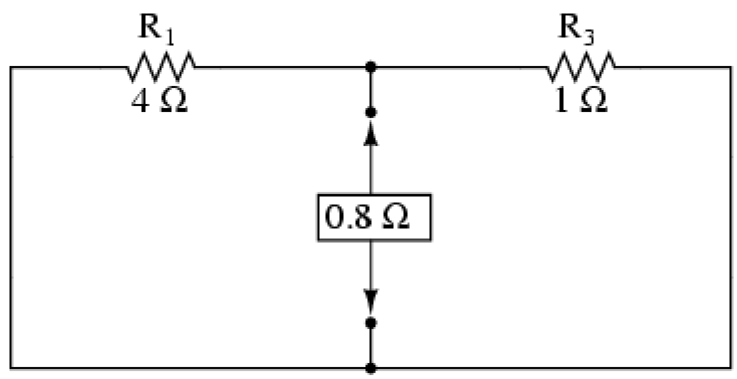


Figure 51: Step 2 - find Norton resistance by shorting the voltages and opening the current sources.

The Norton equivalent circuit of the original circuit now looks like this:

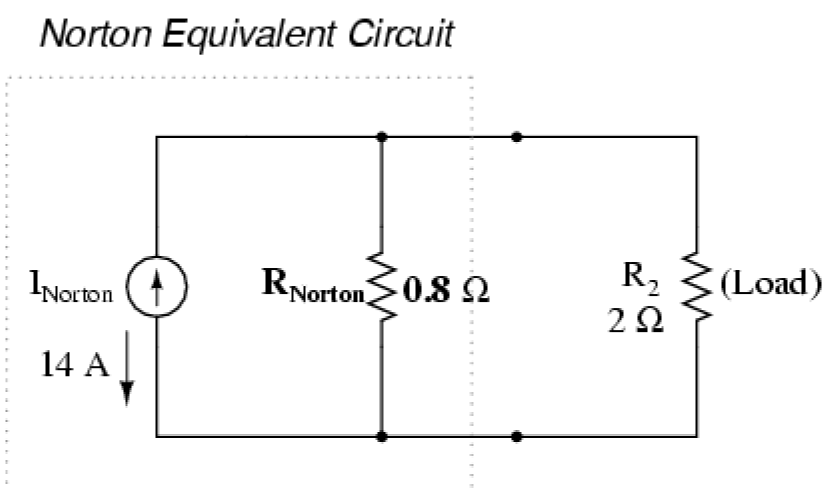


Figure 52: Step 3 – Draw the Norton equivalent circuit.

Reconnect the original load resistance of 2 Ω, and analyze the Norton circuit as a simple parallel arrangement:

	R_{Norton}	R_{Load}	Total	
E	8	8	8	Volts
I	10	4	14	Amps
R	0.8	2	571.43m	Ohms

Figure 53: Step 4 – Analyze voltage and current for R_{Load}

As with the Thevenin equivalent circuit, the only useful information from this analysis is the voltage and current values for R_2; the rest of the information is irrelevant to the original circuit. However, the same advantages as with Thevenin's theorem apply to Norton's as well, allowing for analysis to be performed on changing conditions.

Steps to follow for Norton's Theorem:

1. Find the Norton source current by removing the load resistor from the original circuit and calculating current through a short (wire) jumping across the open connection points where the load resistor used to be.

2. Find the Norton resistance by removing all power sources in the original circuit (voltage sources shorted and current sources open) and calculating total resistance between the open connection points.

3. Draw the Norton equivalent circuit, with the Norton current source in parallel with the Norton resistance. The load resistor re-attaches between the two open points of the equivalent circuit.

4. Analyze voltage and current for the load resistor following the rules for parallel circuits.

Thevenin-Norton Equivalencies

Since Thevenin's and Norton's theorems are two equally valid methods of reducing a complex network down to something simpler to analyze, there must be some way to convert a Thevenin equivalent circuit to a Norton equivalent circuit, and vice versa.

Notice that the procedure for calculating Thevenin resistance is identical to the procedure for calculating Norton resistance: remove all power sources and determine resistance between the open load connection points. As such, Thevenin and Norton resistances for the same original network must be equal. Using the example circuits from the last two sections, we can see that the two resistances are indeed equal and therefore, $R_{Thevenin} = R_{Norton}$.

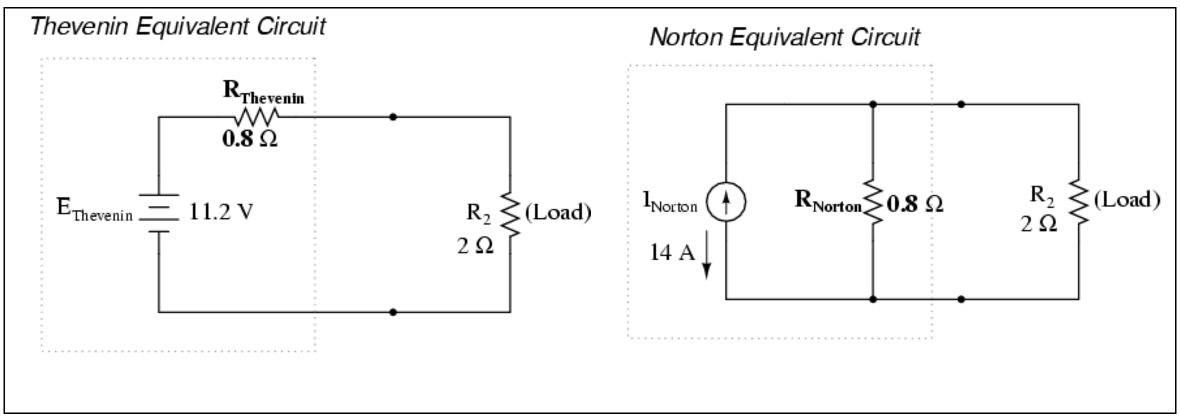


Figure 54: Thevenin and Norton equivalent circuits.

Considering the fact that both Thevenin and Norton equivalent circuits are intended to behave the same as the original network in supplying voltage and current to the load resistor (as seen from the perspective of the load connection points), these two equivalent circuits, having been derived from the same original network should behave identically.

This means that both Thevenin and Norton equivalent circuits should produce the same voltage across the load terminals with no load resistor attached. With the Thevenin equivalent, the open-circuited voltage would be equal to the Thevenin source voltage (no circuit current present to drop voltage across the series resistor), which is 11.2 volts in this case. With the Norton equivalent circuit, all 14 amps from the Norton current source would have to flow through the 0.8 Ω Norton resistance, producing the exact same voltage, 11.2 volts (E=IR). Thus, we can say that the Thevenin voltage is equal to the Norton current times the Norton resistance: $E_{Thevenin} = I_{Norton}R_{Norton}$

So, if we wanted to convert a Norton equivalent circuit to a Thevenin equivalent circuit, we could use the same resistance and calculate the Thevenin voltage with Ohm's Law.

Conversely, both Thevenin and Norton equivalent circuits should generate the same amount of current through a short circuit across the load terminals. With the Norton equivalent, the short-circuit current would be exactly equal to the Norton source current, which is 14 amps in this case. With the Thevenin equivalent, all 11.2 volts would be applied across the 0.8 Ω Thevenin resistance, producing the exact same current through the short, 14 amps (I=E/R). Thus, we can say that the Norton current is equal to the Thevenin voltage divided by the Thevenin resistance:

$$I_{Norton} = \frac{E_{Thevenin}}{R_{Thevenin}}$$

This equivalence between Thevenin and Norton circuits can be a useful tool in itself, as we shall see in the next section.

Millman's Theorem revisited

You may have wondered where we got that strange equation for the determination of Millman voltage across parallel branches of a circuit where each branch contains a series resistance and voltage source:

Millman's Theorem Equation

$$\frac{\frac{E_{B1}}{R_1} + \frac{E_{B2}}{R_2} + \frac{E_{B3}}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \text{Voltage across all branches}$$

Parts of this equation seem familiar to equations we've seen before. For instance, the denominator of the large fraction looks conspicuously like the denominator of our parallel resistance equation. Moreover, the E/R terms in the numerator of the large fraction should give figures for current, Ohm's law being what it is (I=E/R).

With understanding of Thevenin and Norton source equivalencies, it is now possible to understand Millman's equation. What Millman's equation is actually doing is treating each branch (with its series voltage source and resistance) as a Thevenin equivalent circuit and then converting each one into equivalent Norton circuits.

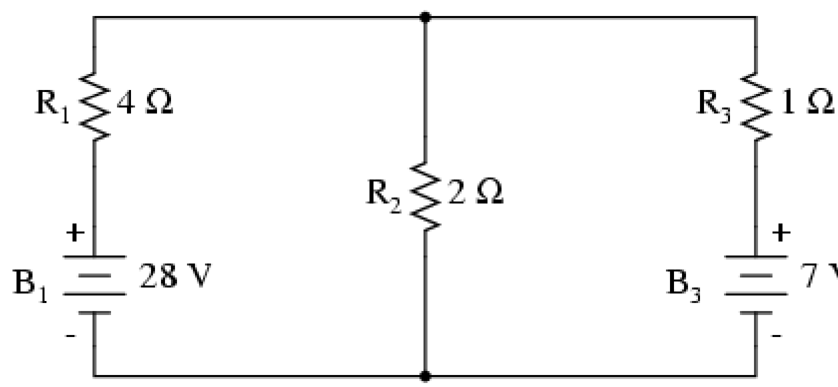


Figure 55: Series-parallel circuit with multiple voltage sources.

Thus, in the circuit above, battery B_1 and resistor R_1 are seen as a Thevenin source to be converted into a Norton source of 7 amps (28 volts / 4 Ω) in parallel with a 4 Ω resistor. The rightmost branch will be converted into a 7 amp current source (7 volts / 1 Ω) and 1 Ω resistor in parallel. The center branch, containing no voltage source at all, will be converted into a Norton source of 0 amps in parallel with a 2 Ω resistor:

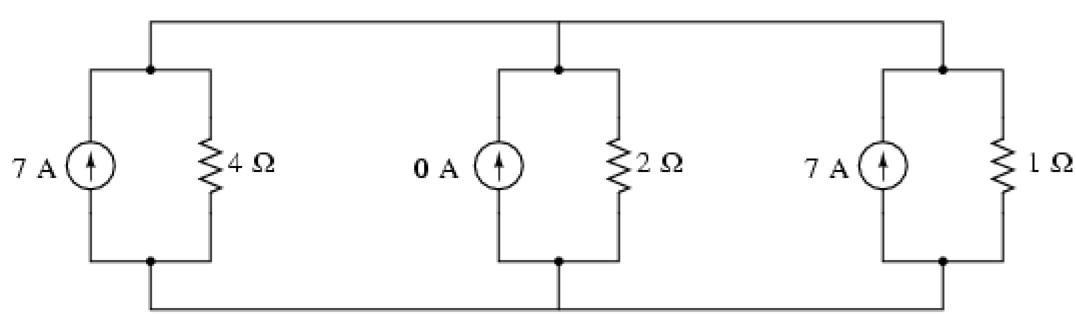


Figure 56: Converted circuit.

Since current sources directly add their respective currents in parallel, the total circuit current will be 7 + 0 + 7, or 14 amps. This addition of Norton source currents is what is being represented in the numerator of the Millman equation:

Millman's Theorem Equation

$$I_{total} = \frac{E_{B1}}{R_1} + \frac{E_{B2}}{R_2} + \frac{E_{B3}}{R_3} \longrightarrow \frac{\mathbf{\frac{E_{B1}}{R_1} + \frac{E_{B2}}{R_2} + \frac{E_{B3}}{R_3}}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

All the Norton resistances are in parallel with each other as well in the equivalent circuit, so they diminish to create a total resistance. This diminishing of source resistances is what is being represented in the denominator of the Millman's equation:

Millman's Theorem Equation

$$R_{total} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} \longrightarrow \frac{\frac{E_{B1}}{R_1} + \frac{E_{B2}}{R_2} + \frac{E_{B3}}{R_3}}{\mathbf{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}}$$

In this case, the resistance total will be equal to 571.43 milliohms (571.43 mΩ). We can re-draw our equivalent circuit now as one with a single Norton current source and Norton resistance:

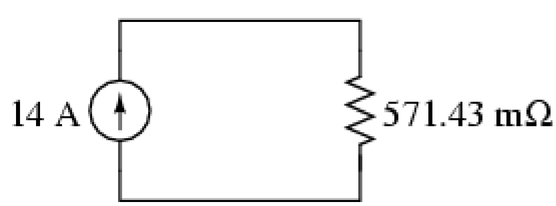


Figure 57: Norton equivalent circuit.

Ohm's Law can tell us the voltage across these two components now (E=IR):

$$E_{total} = (14\ A)(571.43\ m\Omega)$$

$$E_{total} = 8\ V$$

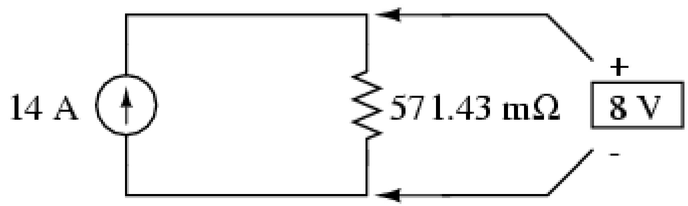


Figure 58: Voltage of equivalent circuit

Let's summarize what we know about the circuit thus far. We know that the total current in this circuit is given by the sum of all the branch voltages divided by their respective currents. We also know that the total resistance is found by taking the reciprocal of all the branch resistance reciprocals. Furthermore, we should be well aware of the fact that total voltage across all the branches can be found by multiplying total current by total resistance (E=IR). All we need to do is put together the two equations we had earlier for total circuit current and total resistance, multiplying them to find total voltage:

Ohm's Law: $I \times R = E$

(total current) x (total resistance) = (total voltage)

$$\frac{E_{B1}}{R_1} + \frac{E_{B2}}{R_2} + \frac{E_{B3}}{R_3} \quad \times \quad \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \text{(total voltage)}$$

. . . or . . .

$$\frac{\frac{E_{B1}}{R_1} + \frac{E_{B2}}{R_2} + \frac{E_{B3}}{R_3}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \textbf{(total voltage)}$$

Figure 59: Summary of Millman's equation

The Millman's equation is nothing more than a Thevenin-to-Norton conversion matched together with the parallel resistance formula to find total voltage across all the branches of the circuit.

Network conversions

In many circuit applications, we encounter components connected together in one of two ways to form a three-terminal network: the “Delta,” or Δ (also known as the “Pi,” or π) configuration, and the “Y” (also known as the “T”) configuration.

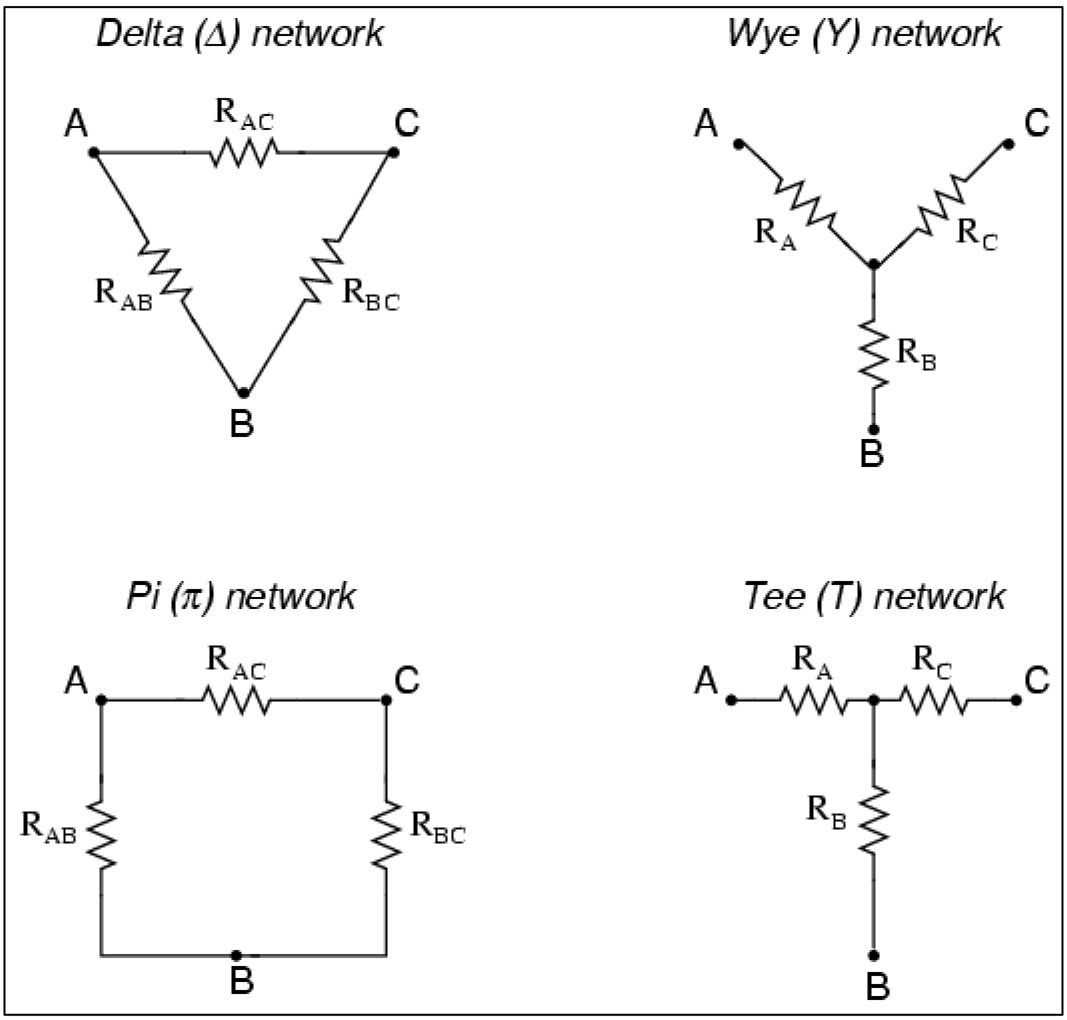


Figure 60: Various network configurations.

It is possible to calculate the proper values of resistors necessary to form one kind of network (Δ or Y) that behaves identically to the other kind, as analyzed from the terminal connections alone. That is, if we had two separate resistor networks, one Δ and one Y, each with its resistors hidden from view, with nothing but the three terminals (A, B, and C) exposed for testing, the resistors could be sized for the two networks so that there would be no way to electrically determine one network apart from the other. In other words, equivalent Δ and Y networks behave identically.

There are several equations used to convert one network to the other:

To convert a Delta (Δ) to a Wye (Y)	*To convert a Wye (Y) to a Delta (Δ)*
$R_A = \frac{R_{AB} R_{AC}}{R_{AB} + R_{AC} + R_{BC}}$	$R_{AB} = \frac{R_A R_B + R_A R_C + R_B R_C}{R_C}$
$R_B = \frac{R_{AB} R_{BC}}{R_{AB} + R_{AC} + R_{BC}}$	$R_{BC} = \frac{R_A R_B + R_A R_C + R_B R_C}{R_A}$
$R_C = \frac{R_{AC} R_{BC}}{R_{AB} + R_{AC} + R_{BC}}$	$R_{AC} = \frac{R_A R_B + R_A R_C + R_B R_C}{R_B}$

Figure 61: Network Conversion Formulas

Δ and Y networks are seen frequently in 3-phase AC power systems, but even then they're usually balanced networks (all resistors equal in value) and conversion from one to the other need not involve such complex calculations. When would the average technician ever need to use these equations?

A prime application for Δ-Y conversion is in the solution of unbalanced bridge circuits, such as shown in Figure 62.

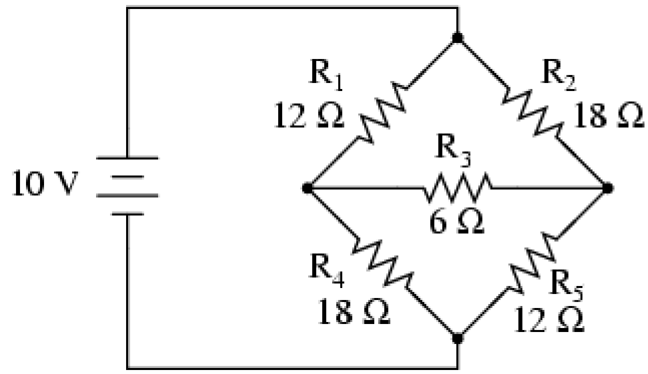


Figure 62: Unbalanced bridge circuit.

Solution of this circuit with branch current or mesh current analysis is fairly involved, and neither the Millman's nor superposition theorems are of any help, since there is only one source of power. Thevenin's or Norton's theorem could be used, treating R_3 as the load.

If we were to treat resistors R_1, R_2, and R_3 as being connected in a Δ configuration (R_{ab}, R_{ac}, and R_{bc}, respectively) and generate an equivalent Y network to replace them, we could turn this bridge circuit into a (simpler) series-parallel combination circuit:

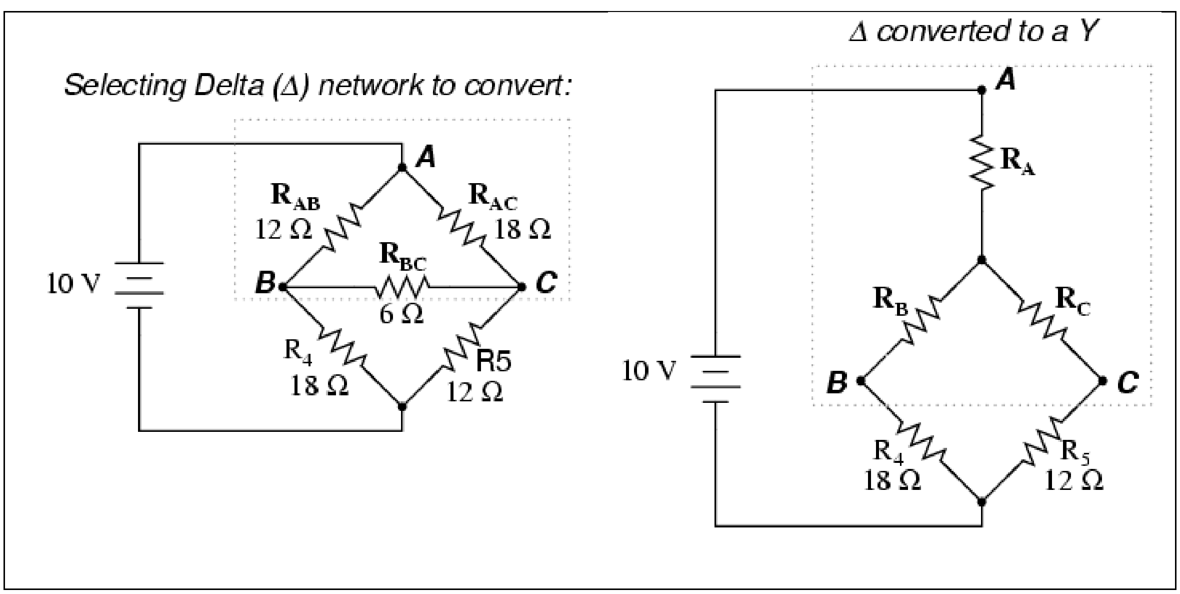


Figure 63: Conversion of unbalanced bridge.

If we perform our calculations correctly, the voltages between points A, B, and C will be the same in the converted circuit as in the original circuit, and we can transfer those values back to the original bridge configuration.

$$R_A = \frac{(12\ \Omega)(18\ \Omega)}{(12\ \Omega) + (18\ \Omega) + (6\ \Omega)} = \frac{216}{36} = \mathbf{6\ \Omega}$$

$$R_B = \frac{(12\ \Omega)(6\ \Omega)}{(12\ \Omega) + (18\ \Omega) + (6\ \Omega)} = \frac{72}{36} = \mathbf{2\ \Omega}$$

$$R_C = \frac{(18\ \Omega)(6\ \Omega)}{(12\ \Omega) + (18\ \Omega) + (6\ \Omega)} = \frac{108}{36} = \mathbf{3\ \Omega}$$

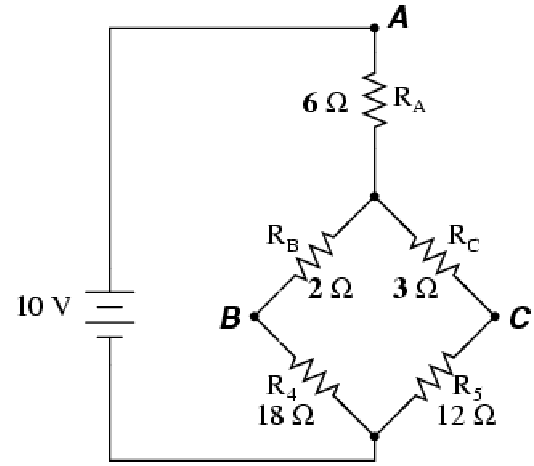


Figure 64: Conversion back to original configuration.

Resistors R_4 and R_5, of course, remain the same at 18 Ω and 12 Ω, respectively. Analyzing the circuit now as a series-parallel combination, we arrive at the following figures:

	R_A	R_B	R_C	R_4	R_5	
E	4.118	588.24m	1.176	5.294	4.706	Volts
I	686.27m	294.12m	392.16m	294.12m	392.16m	Amps
R	6	2	3	18	12	Ohms

	$R_B + R_4$	$R_C + R_5$	$R_B + R_4$ // $R_C + R_5$	Total	
E	5.882	5.882	5.882	10	Volts
I	294.12m	392.16m	686.27m	686.27m	Amps
R	20	15	8.571	14.571	Ohms

Figure 65: Analysis of circuit

We must use the voltage drops figures from the table above to determine the voltages between points A, B, and C, seeing how the add up (or subtract, as is the case with voltage between points B and C):

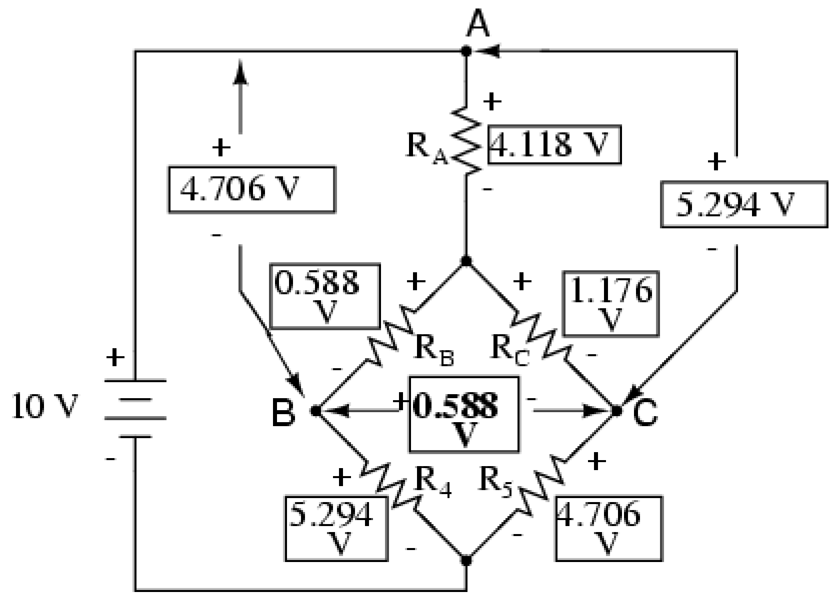


Figure 66: Voltage drops across various points in the circuit.

Now that we know these voltages, we can transfer them to the same points A, B, and C in the original bridge circuit:

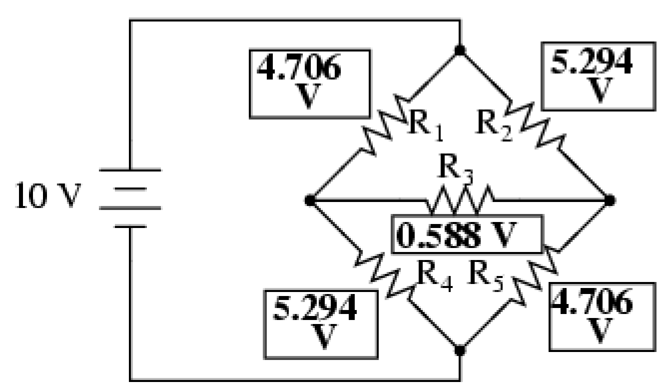


Figure 67: Original bridge circuit with voltage drops.

Additional Resources

Physics Resources

Georgia State University – HyperPhysics

http://hyperphysics.phy-astr.gsu.edu/hbase/hframe.html

Video Resources

Khan Academy – Electricity and magnetism

https://www.khanacademy.org/science/physics/electricity-and-magnetism

References

Kuphaldt, T. (2006). DC Network Analysis. In Lessons in Electric Circuits, Volume I - DC (5th ed., pp. 329 - 390).

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